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Japan***Abstract**

Making use of toric geometry we construct a class of global F-theory GUT models. The base manifolds are blowups of Fano threefolds and the Calabi-Yau fourfold is a complete intersection of two hypersurfaces. We identify possible GUT divisors and construct $SO(10)$ models on them using the spectral cover construction. We use a split spectral cover to generate chiral matter on the **10** curves in order to get more degrees of freedom in phenomenology. We use abelian flux to break $SO(10)$ to $SU(5) \times U(1)$ which is interpreted as a flipped $SU(5)$ model. With the GUT Higgses in the $SU(5) \times U(1)$ model it is possible to further break the gauge symmetry to the Standard Model. We present several phenomenologically attractive examples in detail.

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1 Introduction

Triggered by [1, 2, 3] F-theory has recently received new attention as a natural setup for constructing realistic GUT models within string theory. One of the key properties of the recent F-theory GUT constructions is that it is possible to decouple gravity from the gauge theory and to consider the GUT model locally on the seven brane geometry. After the embedding of a local model into a global geometry the issue of the decoupling limit becomes more delicate and its existence will be a central guideline in our search of global models. Some attractive features of the local models are that they are straightforward to construct and manage to solve many problems that plague GUT models, such as proton decay, GUT-breaking and doublet-triplet splitting, the μ -term problem, and flavor hierarchy elegantly from geometry and with little fine tuning. Given these features it is desirable to see whether it is possible to embed the local models consistently into a global setup of F-theory compactifications on compact Calabi-Yau fourfolds. One first step in this direction was done by so called semi-local models [4, 5, 6, 7, 8] where conditions for the embedding of local models into a Calabi-Yau fourfold have been discussed, and some global features such as fluxes and monodromies have been introduced into the local setup. Global F-theory GUT models have been first constructed in [9, 10, 11, 12]. The authors of [10, 12] have made use of toric geometry to construct and examine Calabi-Yau fourfolds. It turned out that for F-theory GUTs it is natural not to look for fourfolds that are hypersurfaces in some toric ambient space but to go to the sometimes more involved case of complete intersections. Concretely, one looks at complete intersections of two hypersurfaces in a six-dimensional ambient space. One of these equations describes a three dimensional base manifold in a four dimensional projection of the sixfold. The other equation describes the elliptic fibration. The aim of this publication is to systematically construct a class of global models within this framework, to discuss their properties and eventually explicitly construct semi-realistic examples. Since $SU(5)$ F-theory GUTs have already received a lot of attention in the literature, our discussion will focus on $SO(10)$ models. Note however that the geometries we will construct are also perfectly suitable for $SU(5)$ F-theory models.

Let us now give an overview on how we are going to approach the discussion of global models. The first step is to make use of toric geometry to construct a complex three-dimensional base manifold B which is a hypersurface in a toric ambient space. As a starting point we will consider Fano threefolds with one Kähler class. It has been argued in [13] that Fanos are not good base manifolds for F-theory GUTs because they do not allow for a decoupling limit. To remedy this problem we will blow up curves and points inside the Fano threefolds. As a consequence the resulting manifold will in general no longer be Fano. After the construction of a suitable base manifold we go on to search for possible candidates of GUT divisors inside the base. We impose two requirements on the divisor which are important for F-theory GUTs. Firstly, we will look for divisors which are del Pezzo and secondly there should exist a mathematical or at least a physical decoupling limit [13, 12]. The former condition means that the GUT divisor can be shrunk to zero size while the volume of the base manifold remains finite. The latter condition means that we can keep the volume of the GUT divisor finite while the volume of the base manifold becomes infinitely large. Having found a base manifold with a divisor satisfying these elementary requirements we torically construct a Calabi-Yau fourfold which is an elliptic fibration over the base manifold and is characterized in toric geometry by reflexive polyhedra. We note in passing that this toric construction is actually not possible for every base manifold. Having an explicit construction of the Calabi-Yau fourfolds enables us to directly calculate the Euler number of the fourfold. We have compared our results with a formula for the Euler number proposed in [10] and report on a mismatch we find for several examples. We propose possible resolutions for this discrepancy in the Conclusions section.

We then go on to construct $SO(10)$ models in our geometries making use of the spectral cover construction [14, 15, 16, 17]. Originally the spectral cover has been introduced to describe vector bundles in the heterotic string, which has a connection to F-theory via the heterotic/F-theory duality [16]. However, it has been realized in [5, 7] that the spectral cover can also be used in F-theory models without a heterotic dual to locally describe fluxes near the GUT brane. Starting with an elliptic fibration described by a Weierstrass model near the GUT cycle, we extract the terms responsible for the breaking of a E_8 singularity down to a $SO(10)$ singularity. Via the spectral cover picture, these are related to some data of a bundle with structure group in the complement of the GUT group in E_8 which breaks the excess symmetry of E_8 . For the case of $SO(10)$ models one must look at a $SU(4)$ spectral cover. The $SU(4)$ vector bundle V represents the **16**, while the bundle $\wedge^2 V$ represents the **10** of $SO(10)$. Thus we can have a minimal $SO(10)$ GUT model with the **16 16 10** Yukawa coupling. However, it has been shown that the net chirality on the **10** curve vanishes because it forms a double curve [4], which would leave us without a suitable Higgs candidate. In addition, there are not enough degrees of freedom to adjust the number of generations on the matter curves, so the model would not look very realistic. A solution to these problems is to introduce a split spectral cover [18, 8]. Furthermore, having constructed a $SO(10)$ GUT we have yet to show that we can obtain the Standard Model after gauge breaking. Direct breaking from $SO(10)$ to the Standard Model gauge group requires to turn on non-abelian fluxes [3]. Up to now this mechanism has not been explicitly studied. We will make an intermediate step and first break to a model with a $SU(5)$ gauge group by a $U(1)$ flux F_X . Since we have then exhausted the option to break the gauge symmetry by turning on flux, the further gauge breaking will rely on different breaking mechanisms in the GUT model. The natural choice is to break from $SO(10)$ to an $SU(5)$ GUT. However since from the local model construction we have no chirality for the adjoint representation [3], the GUT Higgs **24** which is needed for breaking the $SU(5)$ is absent. Therefore we consider a flipped $SU(5)$ model

[19, 20, 21], (for recent discussion in F-theory, see [22, 23, 24, 25]) where the gauge breaking to the Standard Model can be achieved by the $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ superheavy Higgs fields. We will illustrate our discussion by several examples.

This paper is organized as follows. In Section 2 we will describe the toric construction of the geometries of the global models. Section 3 is devoted to the spectral cover construction. After a brief outline of the construction we will specialize to $SU(4)$ spectral covers and $S(U(3) \times U(1))$ covers which are relevant for $SO(10)$ GUTs. In Section 4 we will give explicit examples of some models. Section 5 is devoted to the phenomenology of the models we have constructed. Conclusions and directions for further research will be given in Section 6. In Appendix A we collect the data of the base manifolds which contain possible GUT divisors. In Appendix B we present the toric data of the Calabi-Yau fourfolds of the models we have worked out in Section 4.

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2 Global toric constructions

In this section we will discuss toric constructions of both the base manifold and the Calabi-Yau fourfold which is an elliptic fibration over the base.

2.1 General idea

In [10] a prescription for constructing F-theory models as a complete intersection of two hypersurfaces has been given. The hypersurface constraints have the following form:

$$P_B(y_i, w) = 0 \quad P_W(x, y, z, y_i, w) = 0 \quad (1)$$

The first constraint describes the threefold base, the second equation the elliptic fibration encoded in the Weierstrass model, which is conveniently written in its Tate form:

$$P_W = x^3 - y^2 + xyz a_1 + x^2 z^2 a_2 + y z^3 a_3 + x z^4 a_4 + z^6 a_6, \quad (2)$$

where the $a_n(y_i, w)$ are sections of K_B^{-n} . The GUT divisor S is specified by $w = 0$.

For the base manifold [10] suggests the following construction. The starting point is a Fano threefold. A simple choice are Fano threefolds which have one Kähler class. These are \mathbb{P}^3 and quadric, cubic and quartic hypersurfaces in \mathbb{P}^4 . The next step is to generate a suitable four cycle where we will wrap the GUT brane. For local models it was argued [1, 2, 3] that, in order to have a well defined local model, the GUT divisor should be a del Pezzo surface. Furthermore, it has been argued in [13] that Fano base manifolds do not allow for

	y_1	y_2	y_3	y_4	y_5	\sum	B
w_1	1	1	1	1	1	5	d

Table 1: Weight vector and degree of the hypersurface equation of $\mathbb{P}^4[d]$.

a decoupling limit in a global F-theory GUT. Following [10, 12] we will blow up points and curves inside the Fano threefolds until the requirements for a GUT model are satisfied. This is discussed in Section 2.2. Most of the data that are relevant for the construction of the GUT model are already encoded in the base manifold. However, in order to discuss global issues like the cancellation of D3 tadpoles it is necessary to have an explicit construction of the fourfold. Furthermore not every base manifold will allow for a toric elliptically fibered Calabi-Yau fourfold whose geometry is encoded in terms of reflexive lattice polytopes. We will discuss this in Section 2.3.

2.2 Toric construction of the base manifold

We will now discuss how to obtain a suitable base manifold for a GUT model by blowing up points and curves inside a Fano threefold. Such constructions were first introduced into the F-theory literature in [10]. Fano threefolds have been classified in [26, 27]. Since we would like to construct our base manifold as a hypersurface in a toric ambient space it is suggestive to start with the following Fanos which are hypersurfaces in \mathbb{P}^4 :

$$\mathbb{P}^4[d] = \{P_d(y_1, \dots, y_5) = 0 | [y_i : \dots : y_5] \in \mathbb{P}^4\} \quad d = 2, 3, 4, \quad (3)$$

where $P_d(y_1, \dots, y_5)$ stands for a polynomial of degree d in the homogeneous coordinates y_i, \dots, y_5 .

It has been argued in [13] that Fanos are not suitable as base manifolds for an F-theory GUT model due to the non-existence of a decoupling limit. For our examples this is trivial to see since the hypersurfaces in (3) only have a single Kähler modulus, and it is therefore impossible to keep the volume of a GUT divisor finite while the size of the base becomes infinitely large, or conversely shrink the GUT divisor to zero size while the volume of the base remains non-zero. For a decoupling limit we need at least two Kähler moduli. We can increase the number of Kähler parameters by blowing up points and curves in the toric ambient space. Such a procedure reduces the number of complex structure moduli and increases the number of Kähler parameters.

The toric data to describe the ambient space and the blowups is encoded in weight vectors [28] which give the homogeneous weights of the coordinates describing the toric variety, or in other words, the weights are the $U(1)$ -charges of the gauged linear sigma model which describes the toric space. For a prescription of how to obtain the toric data from the weight matrices we refer the reader to one of the many books and reviews on toric geometry. The weight matrices are the input data for the package PALP [29] and extensions thereof which can compute all the relevant data we need for the calculations described in this section.

The weight vector for \mathbb{P}^4 is given in table 1, along with some further data: d stands for the degree of the hypersurface equation as in (3), and the next to last entry gives the sum of the weights. We can now blow up curves and points inside the Fano threefold by adding further weight vectors. Blowing up a curve means adding a weight vector as in table 2. Note that due to the symmetry of the ambient space the choice of the weight vector is unique in

	y_1	y_2	y_3	y_4	y_5	y_6	\sum	B
w_1	1	1	1	1	1	0	5	d
w_2	0	0	0	1	1	1	3	

Table 2: Blowup of the first curve.

	y_1	y_2	y_3	y_4	y_5	y_6	\sum	B
w_1	1	1	1	1	1	0	5	d
w_2	1	0	0	0	0	1	2	

Table 3: Blowup of one point.

the sense that we will get the same base equation up to permutations of variables. In table 2 we have not specified the hypersurface degree the base should have in the new weight vector. For $d = (4, 2)$ we recover an example given in [10]. For a point blowup we have to add a weight vector as in table 3. For most interesting models it will not be enough to blow up just one curve or one point. We have systematically searched for models which come from up to three blowups of curves and points inside the Fano threefolds (3).

After the blowups, the resulting base manifolds will in general no longer be Fano. One can ask if these manifolds fit into some mathematically well-defined class of generalizations of Fano threefolds. One possible generalization are the almost Fano manifolds. For a definition see e.g. [30]. An almost Fano threefold is an algebraic threefold \mathcal{F} that has a non-trivial anti-canonical bundle with at least one non-zero section at every point in \mathcal{F} . The Hodge diamond of an (almost) Fano threefold is:

$$\begin{array}{ccccccc}
& & & & 1 & & \\
& & & & 0 & & 0 \\
& & 0 & & b_{1,1} & & 0 \\
0 & & b_{2,1} & & & b_{2,1} & 0 \\
& 0 & & b_{1,1} & & 0 & \\
& & 0 & & 0 & & \\
& & & & 1 & &
\end{array} \tag{4}$$

The Euler number is $\chi = 2(b_{1,1} + 1 - b_{2,1})$. Furthermore, all almost Fano threefolds satisfy [30]:

$$\int_{\mathcal{F}} c_1 c_2 = 24 \tag{5}$$

We discuss the results of our construction of base manifolds, together with a detailed discussion of the restrictions we made, in Appendix A, where we in particular give details about the base manifolds which satisfy the 'almost Fano' condition. For the models we have constructed we observe that the 'almost Fano' property seems to be a necessary but not sufficient condition for the fourfold to be described by a reflexive polytope. It would be interesting to see if observed connection holds more generally and whether it can be put on mathematically solid ground. We plan to further investigate this relation in the future. Section 4 will be devoted to the construction of $SO(10)$ F-theory GUTs from examples among this class.

Having constructed a suitable base manifold, the next step is to find suitable candidates for GUT branes among its divisors. In our search of models we only made two restrictions which are important for the construction of a global GUT model: the GUT divisor should be del Pezzo and there should be a decoupling limit.

We look for candidates for del Pezzo surfaces by identifying divisors which have the same topology as a del Pezzo. Suppose the base manifold has hyperplane class B and is embedded in a toric ambient space with divisors D_i . The total Chern class of a particular divisor S in B is:

$$c(S) = \frac{\prod_i (1 + D_i)}{(1 + B)(1 + S)} \quad (6)$$

A necessary condition for a divisor S to be dP_n is that it must have the following topological data:

$$\int_S c_1(S)^2 = 9 - n \quad \int_S c_2(S) = n + 3 \quad \Rightarrow \quad \chi_h = \int_S \text{Td}(S) = 1, \quad (7)$$

where χ_h is the holomorphic Euler characteristic. Since del Pezzos are Fano twofolds, we have a further necessary condition. The integrals of $c_1(S)$ over all torically induced curves on S have to be positive:

$$D_i \cap S \cap c_1(S) > 0 \quad D_i \neq S \quad \forall D_i \cap S \neq \emptyset. \quad (8)$$

Finding divisors S that have the above properties provides strong evidence that S is indeed a del Pezzo. Whenever we speak about del Pezzo divisors in this presentation we mean divisors which satisfy the conditions above. To be absolutely certain that the divisor is del Pezzo one should explicitly construct it. We did this for the examples we work out in detail in Section 4.

For the model to have a physical decoupling limit, one must be able to tune the Kähler moduli in such a way that the volume of S remains finite while the volume of the base manifold goes to infinity. It is to be contrasted with the mathematical decoupling limit where the GUT divisor shrinks to 0 while the volume of the base remains finite. According to [12] these two decoupling limits may be governed by different vectors in the Kähler cone. In order for the volumes to be positive we have to find a basis K_i of the Kähler cone such that the Kähler form J has the form $J = \sum_i r_i K_i$ with $r_i > 0$. The volumes of the base B and the GUT divisor S are then given by:

$$\text{Vol}(B) = J^3 \quad \text{Vol}(S) = S \cdot J^2 \quad (9)$$

A sufficient condition for the existence of the physical decoupling limit is that the volume of the divisor S does not depend on one or more of the moduli r_i whereas the volume of the base does depend on them. By sending these Kähler moduli to infinity we can achieve infinite volume of the base while keeping the volume of the GUT divisor finite. The mathematical decoupling limit is slightly more difficult to check. There it is necessary to set sufficiently many Kähler parameters to zero such that $\text{Vol}(S) = 0$ while still having terms in $\text{Vol}(B)$ which are independent of the parameters we have set to zero. In order to perform this analysis we need the triple intersection numbers of the divisors, restricted to the base B , and a basis of the Kähler cone¹. This data can be extracted from the weight matrices and the divisor specifying the hypersurface B . The necessary calculations can be done with help of an extended version

¹We constructed the Kähler cone for the toric ambient space. We did not take into account that different phases of the toric ambient space (triangulations) can lead to equivalent phases on the hypersurface, which would yield a larger Kähler cone.

of the package PALP [29]. In Appendix A we list, within our class of examples, the del Pezzo divisors that satisfy at least one of the decoupling conditions.

Having identified a 'nice' del Pezzo inside the base manifold we can now specify the gauge theory we would like to have. For $SO(10)$ models, the a_i s of the Weierstrass equation (2) have to degenerate in a certain way [31]. According to Kodaira's classification of singularities of elliptic curves [32] and Tate's algorithm [33], they have to have the following form²,

$$a_1 = b_5 w^1 \quad a_2 = b_4 w^1 \quad a_3 = b_3 w^2 \quad a_4 = b_2 w^3 \quad a_6 = b_0 w^5, \quad (10)$$

where the b_i s are sections of some appropriate line bundle over B that have at least one term independent of w . In this description matter curves and Yukawa couplings are located at the subsets where the discriminant of (2) has an order one and an order two enhancement, respectively. The order refers here to the vanishing power of the discriminant in the vicinity of S . The matter curves for the $SO(10)$ models are at

$$b_3 = 0 \quad \mathbf{10} \text{ matter} \quad b_4 = 0 \quad \mathbf{16} \text{ matter}, \quad (11)$$

and the Yukawa couplings are at

$$b_3 = 0 \cap b_4 = 0 \quad E_7 \text{ Yukawas} \quad b_2^2 - 4b_0 b_4 = 0 \cap b_3 = 0 \quad SO(14) \text{ Yukawas}. \quad (12)$$

See Section 3.1 for a more detailed discussion. Now we can compute the genus (or respectively, the Euler number) of the generic matter curves and the number of their intersections (Yukawa couplings). The genera of the matter curves can be computed via their first Chern classes. The number of Yukawa couplings is given by the triple intersection of the divisors of the curves with the GUT brane. In practice this is done by expressing c_1 of the curves and the b_i in terms of toric divisors and making use of their triple intersections.

2.3 Fourfold

Having constructed a threefold base B we obtain a Calabi-Yau fourfold by fibering a torus over it. Our aim is to construct a complete intersection (1) of two hypersurfaces in a six dimensional toric manifold which describes this situation. One equation should define the base manifold in a four dimensional projection of the toric sixfold. The other one, given in (2), is the Weierstrass equation which gives us the torus fiber. Since the Weierstrass equation (2) has to be a well defined equation and the a_i s are sections of certain line bundles over the base, also the fiber coordinates x and y have to transform non-trivially over B . To obtain a Calabi-Yau fourfold it turns out that x and y have to be sections of K_B^{-2} and K_B^{-3} , respectively.

A complete intersection Calabi-Yau of codimension r in a toric variety \mathbb{P}_Σ is described by r equations $f_i = 0$ where the f_i are sections of line bundles whose support polytopes [34] are the Newton polytopes Δ_i of f_i and Σ is the fan³ over the faces of a lattice polytope Δ° . By

²For an $SU(5)$ model the structure would be:

$$a_1 = b_5 \quad a_2 = b_4 w^1 \quad a_3 = b_3 w^2 \quad a_4 = b_2 w^3 \quad a_6 = b_0 w^5$$

³The fan Σ is not to be confused with the matter curves which we will also denote by Σ in the following sections.

the adjunction formula we obtain a Calabi-Yau if the Minkowski sum $\Delta = \Delta_1 + \dots + \Delta_r$ is dual to Δ° in the sense that the inequality $\langle \Delta, \Delta^\circ \rangle \geq -1$ is saturated. In particular, Δ and Δ° is a reflexive pair of lattice polytopes. We will restrict our attention to the case where Δ° admits a nef partition into a convex hull of lattice polytopes $\nabla_{i \leq r}$ [35], whose combinatorics is summarized by

$$\begin{aligned} \Delta &= \Delta_1 + \dots + \Delta_r & \Delta^\circ &= \langle \nabla_1, \dots, \nabla_r \rangle_{\text{conv}} \\ (\nabla_n, \Delta_m) &\geq -\delta_{nm} & \\ \nabla^\circ &= \langle \Delta_1, \dots, \Delta_r \rangle_{\text{conv}} & \nabla &= \nabla_1 + \dots + \nabla_r \end{aligned} \tag{13}$$

The Batyrev-Borisov mirror construction [35] exchanges Δ 's with ∇ 's so that the mirror manifold is described by equations whose Newton polytopes are ∇_i . While we have no need for mirror symmetry in the present context, the advantage of this class of complete intersections is that combinatorial formulas for the “string theoretic” Hodge numbers have been derived [36, 37] and implemented in PALP [29].

Calabi-Yau 4-folds in toric varieties in general may have terminal singularities, which are inherited from the ambient space. In order to resolve as many singularities as possible the first step is to choose a maximal projective crepant resolution [38], where crepant means that we do not change the canonical class. In combinatorial terms this means that we choose a maximal coherent triangulation of Δ° and take for Σ the fan over the faces of that triangulation. Since $\mathbb{P}_\Sigma = \bigcup U_\sigma$ is covered by the affine patches U_σ for $\sigma \in \Sigma$ every singularity is located in some U_σ and the maximal dimension of a component of the singular locus is equal to the minimal dimension of a singular cone. For reflexive polytopes the lattice distance of the interior point to every facet of Δ° is one so that U_σ is non-singular if and only if σ is the cone over a simplex of lattice volume 1. Since all triangles of a maximal triangulation of a polygon have minimal volume, the highest-dimensional singularities of \mathbb{P}_Σ come from 4-dimensional cones and therefore have codimension 4. For a Calabi-Yau k -fold X of codimension r the solution set of the r equations generically avoids singular sets of dimension smaller than r so that X is generically smooth for $k \leq 3$. Depending on the choice of triangulation, terminal singularities of dimension $k - 4$ are possible, however, for $k > 3$, and no triangulation might exist for which X is smooth. Therefore, independently of the codimension, smoothness of a CY 4-fold needs to be checked by computing the volumes of the 4-dimensional cones $\sigma \in \Sigma^{(4)}$. Note that string theoretic Hodge numbers are well-defined and independent of the triangulation even in the singular case [36].

While fibration structures depend on the intersection ring and therefore on the (computationally expensive) resolution of singularities, toric fibrations, which are inherited from morphisms of the ambient space, have the advantage of being visible as reflexive plane sections Δ_f° of Δ° [39, 40, 41]. Generically the codimension of the fiber is equal to the codimension r of the Calabi-Yau because it is described by the same (number of) equations, while the base is a toric variety \mathbb{P}_{Σ^B} whose rays are the images of $\Sigma^{(1)}$ under projection along the fiber polytope Δ_f° . It may happen, however, that the complete fiber polytope is contained in one of the ∇_i of the nef partition [42]. Then the remaining equations $f_j = 0$ do not contain the coordinates of the fiber polytope and constrain the base, while the generic fiber becomes a hypersurface. Since we want a fibration in Weierstrass form we are interested in precisely this situation with Δ_f being the Weierstrass triangle.

A transparent way to describe the polytopes is in terms of their coordinate-independent weight matrices [28] (i.e. charges of the GLSM, called “combined weight systems” in [29]).

The fibration is then visible, up to permutations of the columns, as a degree 6 weight vector $(1, 2, 3, 0, \dots, 0)$ in a block form with the weight matrix of the base and the choice of the remaining integers in the columns supporting the Weierstrass fiber define the fibration. It is, of course, a very selective condition that the resulting 6-dimensional polytope is reflexive and admits a nef-partition of the required form. In Appendix A we will indicate which of the base manifolds satisfy these constraints.

For our purposes we now specialize to the case of a complete intersection of two hypersurfaces, i.e. $r = 2$ in (13). The hypersurface constraints can be reconstructed from the toric data as follows:

$$f_m = \sum_{w_k \in \Delta_m} c_k^m \prod_{n=1}^2 \prod_{\nu_i \in \nabla_n} x_i^{\langle \nu_i, w_k \rangle + \delta_{mn}} \quad m, n = 1, 2, \quad (14)$$

where the c_k^m are complex structure parameters. So far we have only specified a generic elliptic fibration over the base manifold. In order to define a specific GUT model we also have to specify the GUT group. We can realize this torically by dropping all the monomials from (Δ_1, Δ_2) which do not have the GUT group specific vanishing degrees as determined by the Tate classification. This procedure amounts to choosing a very non-generic complex structure. This may entail that the Calabi-Yau may not miss the singularities of the toric ambient space. A detailed discussion of these issues is beyond the scope of this article. Note however that the discrepancy we find for several examples between the Euler numbers computed for the fourfolds using toric methods and those calculated a by formula given in [10] may be due to additional singularities away from the GUT surface. We plan to return to these issues in future work.

The Tate form (2) implies that the a_n appear in the monomials which contain z^n . We can isolate these monomials by identifying the vertex ν_z in (∇_1, ∇_2) that corresponds to the z -coordinate. All the monomials that contain z^r are then in the following set:

$$A_r = \{w_k \in \Delta_m : \langle \nu_z, w_k \rangle - 1 = r\} \quad \nu_z \in \nabla_m, \quad (15)$$

where Δ_m is the dual of ∇_m , which denotes the polytope containing the z -vertex. The polynomials a_r are then given by the following expressions:

$$a_r = \sum_{w_k \in A_r} c_k^m \prod_{n=1}^2 \prod_{\nu_i \in \nabla_n} y_i^{\langle \nu_i, w_k \rangle + \delta_{mn}} \Big|_{x=y=z=1} \quad (16)$$

Since we are only interested in the coefficients a_n in the Tate form, we have set those y_i which correspond to (x, y, z) to 1.

Next we have to select the subset of the a_n which is compatible with the GUT group. This amounts to fixing a particular gauge group on the GUT brane S defined by $w = 0$. The order of vanishing of the a_n is dictated by the power of w in the monomial. The Tate classification then implies that $(a_1, a_2, a_3, a_4, a_6)$ can be factored as $(b_5 w^{k_1}, b_4 w^{k_2}, b_3 w^{k_3}, b_2 w^{k_4}, b_0 w^{k_6})$ for some positive integers k_i . Since the sections b_i can still depend on w , we only drop monomials whose w -powers are smaller than indicated by the Tate classification.

Using this procedure we get polyhedra $(\tilde{\Delta}_1, \tilde{\Delta}_2)$ which contain fewer points than (Δ_1, Δ_2) . The duals $(\tilde{\nabla}_1, \tilde{\nabla}_2)$ will then contain more points. The crepant resolution of this new fourfold

probably resolves all the GUT singularities. However again the issue with terminal singularities may arise. In this way we have explicitly constructed the toric Calabi-Yau fourfold which encodes the Euler number for the D3 brane tadpole cancellation condition to a particular F-theory GUT model.

Note that removing points from a polytope is a quite delicate procedure that may destroy certain features such as reflexivity. In our Model 4 discussed in Section 4 and Appendix B reflexivity is lost for one of the dP_5 s after imposing the $SO(10)$ Weierstrass model.

3 Spectral cover

Originally the spectral cover construction has been introduced in the context of heterotic string theory as a means to describe stable bundles on elliptically fibered threefolds [16, 17, 43, 18]. Via heterotic/F-theory duality it is natural to use this construction also in F-theory models with heterotic duals. In fact this structure is intrinsic to eight dimensional supersymmetric Yang-Mills theory. Therefore it was realized that the spectral cover construction does not only apply to models with a heterotic dual, see [4] and [7]. In type II language its data can be interpreted as B-branes in an auxiliary non-compact Calabi-Yau threefold X . The authors of [10, 12] found that the spectral cover also seems to have some validity beyond the local limit. In [10] a formula based on the spectral cover for the Euler number of the fourfold has been given which has been shown to match with the direct calculation of the Euler number using toric geometry. In the following sections we will put this formula to the test in several examples.

There are two more equivalent ways to specify an eight dimensional supersymmetric gauge theory: ALE fibration over the GUT surface and G -flux and the Higgs bundle picture. We will not elaborate on them here but it may be useful to switch between these pictures. For further details see [7, 8].

3.1 $SO(10)$ models

At the end of Section 2.2 we have already considered $SO(10)$ singularities (gauge groups) along a surface S ($w = 0$). In this section we will give more details about this. The form of the degeneration of the elliptic fiber of the CY_4 on S was given by:

$$y^2 = x^3 + b_5 w x y + b_4 w x^2 + b_3 w^2 y + b_2 w^3 x + b_0 w^5, \quad (17)$$

where now we are looking at the patch $z \neq 0$. By completing the square we can rewrite the above equation into the Weierstrass form,

$$\tilde{y}^2 = \tilde{x}^3 + f \tilde{x} + g, \quad (18)$$

where f and g are sections of K_B^{-4} and K_B^{-6} , respectively, with the following expansion in w :

$$f = \sum f_m w^m = -\frac{b_4^2}{3} w^2 + (b_2 + \frac{b_3 b_5}{2} - \frac{b_4 b_5^2}{6}) w^3 - \frac{b_5^4}{48} w^4, \quad (19)$$

$$g = \sum g_n w^n = \frac{2 b_4^3}{27} w^3 + \frac{1}{36} (9 b_3^2 - 12 b_2 b_4 - 6 b_3 b_4 b_5 + 2 b_4^2 b_5^2) w^4 \\ + (b_0 - \frac{6 b_2 b_5^2}{72} - 3 b_3 b_5^3 + b_4 b_5^4) w^5 + \frac{b_5^6}{864} w^6. \quad (20)$$

For the discriminant, $\Delta = 4f^3 + 27g^2$, we obtain,

$$\begin{aligned}\Delta = & b_3^2 b_4^3 w^7 + \frac{1}{16} (27 b_3^4 + 16 b_4^2 (-b_2^2 + 4 b_0 b_4) - 4 b_3 b_4 b_5 (9 b_3^2 + 4 b_2 b_4) \\ & + 8 b_3^2 b_4 (-9 b_2 + b_4 b_5^2)) w^8 + \mathcal{O}(w^9).\end{aligned}\quad (21)$$

Therefore we can now really identify the singularity enhancements stated in Section 2.2:

- First order enhancement to an E_6 singularity along the curve $b_4 = 0$ —**16** matter curve.
- First order enhancement to an $SO(12)$ singularity along the curve $b_3 = 0$ —**10** matter curve.
- Second order enhancement to an E_7 singularity over the intersection points of $b_4 = 0$ and $b_3 = 0$ —**16 16 10** Yukawa coupling points.
- Second order enhancement to an $SO(14)$ singularity over the intersection points of $b_3 = 0$ and $b_2^2 - 4b_0b_4 = 0$ —**10 10 16** Yukawa coupling points.

The reason for the labeling of the curves and Yukawa couplings comes from group theory and the fact that the singularities of the fibration are directly related to the gauge theory on S [31, 44]. For an E_8 singularity we would have E_8 SYM on S , and hence, matter in the adjoint representation of E_8 . If we now deform the singularity to $SO(10)$ we obtain the following breaking pattern:

$$\begin{aligned}E_8 & \supset SU(4) \times SO(10) \\ \mathbf{248} & \rightarrow (\mathbf{15}, \mathbf{1}) + (\mathbf{1}, \mathbf{45}) + (\mathbf{6}, \mathbf{10}) + (\mathbf{4}, \mathbf{16}) + (\overline{\mathbf{4}}, \overline{\mathbf{16}}).\end{aligned}\quad (22)$$

The **45** is the adjoint representation of the $SO(10)$ gauge theory on S . The **16**, $\overline{\mathbf{16}}$ and the fundamental representation arise at the curves of higher singularity. To see that we have a **10** matter curve we can use the IIB picture. In type IIB we obtain a $SO(12)$ gauge group if we place six D-branes on top of the orientifold plane. If we now tilt one of the branes we reduce the gauge theory on the O-plane to $SO(10)$ and get matter in the fundamental representation of $SO(10)$ along the intersection of the tilted brane and the O-plane. For the **16** matter curve we have to use group theory since one does not have a IIB analogue at hand. For the adjoint of E_6 we observe the following breaking pattern:

$$\begin{aligned}E_6 & \supset U(1) \times SO(10) \\ \mathbf{78} & \rightarrow \mathbf{1}_0 + \mathbf{45}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3,\end{aligned}\quad (23)$$

where **45** is again the adjoint representation of the $SO(10)$ gauge theory on S . Hence, **16** matter arises at the E_6 enhancement. To see the labeling of the Yukawas, one has to look at the cubic coupling of the adjoints of E_7 and $SO(14)$, respectively, and its breaking to $SO(10)$.

Now we go from the global picture to the local picture in the vicinity of S . We do this by extracting an ALE-fibration from (17). First we 'localize' the b_i s by removing the w -dependent part such that they become sections of bundles on S . From now on we will denote by b_i the 'localized' b_i s which are sections of the following bundles:

$$b_i \in \Gamma(K_S^{-(6-i)} \otimes \mathcal{N}_{S|B}) \quad i = 0, \dots, 4, \quad (24)$$

and $b_5 \in \Gamma(K_S^{-1})$. Then like in [7], we assign the scaling dimensions $(\frac{1}{3}, \frac{1}{2}, \frac{1}{5})$ to (x, y, w) in (17) such that the terms of order one give the E_8 singularity. If we drop all terms of scaling dimension greater than one we are left with

$$y^2 = x^3 + b_4 w x^2 + b_3 w^2 y + b_2 w^3 x + b_0 w^5, \quad (25)$$

where the terms with dimension less than one are relevant deformations of the E_8 singularity. Hence we see that b_5 does not appear in the local construction. This we perceived already in our analysis above where it did not show up in the singularity considerations on S . Since the intersection matrix of the two-cycles introduced by the partial resolutions of the E_8 singularity by relevant deformations is minus the Cartan matrix of A_3 , we can equivalently describe these resolutions by an A_3 (SU_4) singularity that is completely resolved. Thus, we obtain an \widehat{SU}_4 ALE-fibration:

$$y^2 = x^2 + b_0 s^4 + b_2 s^2 + b_3 s + b_4, \quad (26)$$

where s is a coordinate on K_S . As explained in [7] the ALE fibration and the Higgs bundle are equivalent descriptions for the eight dimensional gauge theory on S . The information about the Higgs bundle is encoded in the spectral cover. Since this is a more appropriate description for our purpose, we elaborate on it in the next section.

3.2 $SU(4)$ spectral cover

The spectral cover was introduced in heterotic string theory to encode the breaking of the E_8 gauge symmetry to some gauge group H . It characterizes all the information about the bundle with structure group G ($\subset E_8$) that is the commutant of H in E_8 . In our case, we have an $SO(10)$ gauge group on S , and thus G is $SU(4)$. Since we also arrived at the $SU(4)$ in our above reasoning we have convincing evidence that the spectral cover is applicable, locally in the vicinity of the GUT divisor S , even for F-theory models without a heterotic dual.

The starting point of the construction is the local Calabi-Yau threefold $X = (K_S \rightarrow S)$ [7]. It is convenient to compactify this non-compact CY to

$$\bar{X} = \mathbb{P}(K_S \oplus \mathcal{O}_S) \quad \text{with} \quad \pi : \bar{X} \rightarrow S, \quad (27)$$

where $[u : v]$ are the homogeneous coordinates of the fiber. \bar{X} is a compact space but no longer Calabi-Yau. The divisor classes of the sections σ and σ_c are $\sigma \sim u$ and $\sigma_c \sim v$ in \bar{X} , respectively, where σ_c denotes the section at infinity. As in the non-compact case S is given by the zero section, $\sigma = 0$. Since $\sigma_c \cdot \sigma = 0$ we have:

$$\sigma \cdot \sigma = -\sigma \cdot c_1(S). \quad (28)$$

The first Chern class of \bar{X} is $c_1(\bar{X}) = 2\sigma_c = 2(\sigma + c_1(S))$. The $SU(4)$ spectral cover is now given by the hypersurface,

$$b_0 s^4 + b_2 s^2 + b_3 s + b_4 = 0, \quad (29)$$

inside \bar{X} where s is $u/v \sim c_1(S)$ and the b_i s are the sections from equation (26). Together with the projection π of \bar{X} this induces a fourfold cover of S . For later convenience we denote the divisor $b_0 = 0$ on S by η . This gives us the following relations between the divisors on S :

$$b_i \sim \eta - i c_1(S) \sim (6 - i)c_1(S) - t \quad i = 0, \dots, 4, \quad (30)$$

where $-t$ is a section of $\mathcal{N}_{S|B}$. From equation (29) and (30) we obtain

$$[C_V] = 4\sigma + p_4^*\eta \quad (31)$$

for the divisor class of the spectral cover, where $p_4: C_V \rightarrow S$ denotes the projection from the cover to the GUT surface and V denotes the fundamental representation of G .

As mentioned in the beginning of this section, from a type II perspective the spectral cover can be interpreted as an auxiliary flavor brane. In type II string theory (fundamental) matter arises at the intersection of two branes. Hence, the matter curve(s) in \bar{X} are given by the intersection of C_V with $\sigma = 0$. The corresponding curve Σ_V ($\Sigma_{\mathbf{16}}$) on S can be calculated by

$$[C_V] \cdot \sigma = (4\sigma + \pi^*\eta) \cdot \sigma = \sigma \cdot \pi^*(\eta - 4c_1), \quad (32)$$

which implies that $\Sigma_{\mathbf{16}} \sim \eta - 4c_1$. This is in accord with our singularity analysis above because $b_4 \sim \eta - 4c_1$. Since b_4 is the product of the four roots of (29), denoted by λ_i in the following, $\Sigma_{\mathbf{16}}$ is determined by $\lambda_i = 0$. Naïvely one would guess that one obtains four distinct curves from the four solutions $\lambda_i = 0$, but this is of course not true. This happens only in the case where the polynomial b_4 factors, as we will see in Section 3.3. The reason for this is that the λ_i s are multivalued functions. So monodromies connect the λ_i s, and only when b_4 splits also the monodromy group splits.

As mentioned above the E_8 symmetry is broken by a VEV of the two cycles in the ALE fibration on S . Locally the G flux of the ALE fibration over S can be encoded in terms of the spectral line bundle \mathcal{L} along the spectral cover C_V . Since C_V is a four cover of S , after pushing forward \mathcal{L} one can define a rank four vector bundle $V = p_{4*}\mathcal{L}$. The first Chern class of V has to vanish because we have an $SU(4)$ cover, which implies:

$$0 = c_1(p_{4*}\mathcal{L}) = \pi_*(\mathcal{L} - \frac{1}{2}r), \quad (33)$$

where r is the ramification divisor $r = \pi^*c_1(S) - c_1(C_V)$. If we assume:

$$c_1(\mathcal{L}) = \frac{1}{2}r + \gamma_u, \quad (34)$$

then $\pi_*\gamma_u = 0$. γ_u is the universal flux and for the $SU(4)$ cover C_V we get:

$$\gamma_u = C_V \cdot (4\sigma - \pi^*(\eta - 4c_1)) = 4\Sigma_V - \pi^*(\eta - 4c_1(S)). \quad (35)$$

The net chirality for fermions on the curve $\Sigma_{\mathbf{16}}$ is given by [45, 46, 4]:

$$n_{\mathbf{16}} - n_{\overline{\mathbf{16}}} = \int_{\Sigma_V} \gamma = -\eta \cdot (\eta - 4c_1(S)). \quad (36)$$

Up to now we have only discussed the $\mathbf{16}$ matter curve. The $\mathbf{10}$ curve can be obtained from a $\wedge^2 V$ bundle. In terms of the solutions of the spectral cover (29) the roots associated to $\wedge^2 V$ are given by $\lambda_i + \lambda_j = 0$. Thus the spectral surface $C_{\wedge^2 V}$ can be represented as:

$$C_{\wedge^2 V}: b_0^2 \prod_{i < j} (s + \lambda_i + \lambda_j) = b_0^2 s^6 + 2b_0 b_2 s^4 + (b_2^2 - 4b_0 b_4) s^2 - b_3^2. \quad (37)$$

Therefore $b_3 = 0 \cap s = 0$ defines a $\mathbf{10}$ curve. Using the formula given in [43], or from (37), it would be natural to write $C_{\wedge^2 V} = 6\sigma + 2\pi^*\eta$. We denote the curve supporting the $\mathbf{10}$

representation by $\Sigma_{\mathbf{10}} = C_{\wedge^2 V} \cdot \sigma$. However usually $C_{\wedge^2 V}$ is singular, therefore we would like to construct a resolved cover via C_V . Consider the intersection of C_V and its image under an involution τ . Then one can make the following decomposition [43]:

$$\tau C_V \cap C_V = C_V \cdot \sigma + C_V \cdot 3\sigma_c + D. \quad (38)$$

Here σ is the zero section and $3\sigma_c$ is the trisection intersecting the fiber at the three non-trivial points. D is contained in X . It is the double cover of the support curve $\Sigma_{\wedge^2 V}$ with the map $\pi_D : D \rightarrow \Sigma_{\wedge^2 V}$ and it can be written as $D = C_V \cap (C_V - \sigma - 3\sigma_c)$.

Since the spectral surface $C_{\wedge^2 V}$ forms a double curve, we need to resolve this double-curve singularity. After blowing up the singularities we can define a degree two cover $\tilde{\Sigma}_{\wedge^2 V}$ with a projection $\tilde{\pi}_D : D \rightarrow \tilde{\Sigma}_{\wedge^2 V}$ [4]. The **10** curve is then obtained from $D \cdot \sigma$ which implies the class of $\Sigma_{\mathbf{10}}$ is [4]:

$$\Sigma_{\mathbf{10}} = -3c_1 + \eta. \quad (39)$$

After further resolving the codimension-2 singularities along $C_{\wedge^2 V}$ we obtain the following expression for the chirality for $\Sigma_{\mathbf{10}}$ [4]:

$$\chi(\wedge^2 V) = \int_{\tilde{\Sigma}_{\wedge^2 V}} \tilde{\pi}_{D*} \gamma = \int_{\Sigma_{\wedge^2 V}} \pi_{D*} \gamma = 0. \quad (40)$$

Thus, the net chirality of the **10** curve always vanishes for the $SU(4)$ cover, which is not very promising for the $SO(10)$ GUT model construction, because of the absence of the **10** Higgs. Even if we turn on a flux in the bulk to break the $SO(10)$ gauge group to $SU(5)$, the structure is not abundant enough to realize a realistic model. Therefore in what follows we will consider splitting the spectral cover to enrich this construction.

3.3 Spectral cover splitting

Trying to build $SO(10)$ GUT models from $SU(4)$ cover we encounter the problem that the chirality of the **10** curve is generically zero. A possible way to solve this problem is to factorize the spectral curve in order to obtain chiral matter on each of the individual curves. A convenient choice is the $(1, 3)$ factorization. This means that we split the cover group $SU(4)$ into $S[U(3) \times U(1)]^4$. The total group structure of the E_8 breaking is:

$$\begin{aligned} E_8 &\supset SU(4)_\perp \times SO(10) \supset S[U(3) \times U(1)] \times SO(10) \\ \mathbf{248} &\rightarrow (\mathbf{15}, \mathbf{1}) + (\mathbf{1}, \mathbf{45}) + (\mathbf{6}, \mathbf{10}) + (\mathbf{4}, \mathbf{16}) + (\bar{\mathbf{4}}, \bar{\mathbf{16}}) \\ &\rightarrow (\mathbf{8}, \mathbf{1})_0 + (\mathbf{3}, \mathbf{1})_{-4} + (\bar{\mathbf{3}}, \mathbf{1})_4 + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{45})_0 + (\mathbf{3}, \mathbf{10})_2 \\ &\quad + (\bar{\mathbf{3}}, \mathbf{10})_{-2} + (\mathbf{1}, \mathbf{16})_3 + (\mathbf{3}, \mathbf{16})_{-1} + (\mathbf{1}, \bar{\mathbf{16}})_{-3} + (\bar{\mathbf{3}}, \bar{\mathbf{16}})_1 \end{aligned} \quad (41)$$

The cover C_V can be factorized as $C_V \rightarrow C^{(1)} + C^{(3)}$. With the homogeneous coordinates $[u : v]$ of \mathbb{P}_1 the polynomial (29) turns out to be:

$$b_0 u^4 + b_1 u^3 v + b_2 u^2 v^2 + b_3 u v^3 + b_4 v^4 = (a_0 u^3 + a_1 u^2 v + a_2 u v^2 + a_3 v^3)(d_0 u + d_1 v) = 0, \quad (42)$$

⁴The reason we do not write $SU(3) \times U(1)$ is that we only demanded $c_1(V) = 0$ for the $SU(4)$ vector bundle before the cover is split. The $U(1)$ can be split off as a gauge symmetry if one extends the spectral cover factorization to a global restriction of the Tate model [47]. In this case, $E_8 \rightarrow SO(10) \times [SU(3) \times U(1)]$ and the gauge group of the GUT model can be taken as $SO(10) \times U(1)$.

where

$$b_0 = a_0 d_0, \quad b_1 = a_1 d_0 + a_0 d_1 = 0, \quad b_2 = a_2 d_0 + a_1 d_1, \quad b_3 = a_3 d_0 + a_2 d_1, \quad b_4 = a_3 d_1. \quad (43)$$

Since the factorization is not unique, we make the following assumption for the corresponding class:

$$[d_0] = c_1 + \xi, \quad [d_1] = \xi, \quad [a_i] = \eta - (i+1)c_1 - \xi. \quad (44)$$

Therefore we can write the class of the split cover as:

$$C_V = C^{(1)} + C^{(3)} = (\sigma + \pi^*(c_1 + \xi)) + (3\sigma + \pi^*(\eta - c_1 - \xi)) \quad (45)$$

As mentioned above the **10** curve is obtained from the bundle $\wedge^2 V$, i.e. from $C_V \cap \tau C_V$ with the involution $\tau : v \rightarrow -v$. After splitting the cover we get:

$$C_V \cap \tau C_V \rightarrow C^{(1),(1)} + C^{(1),(3)} + C^{(3),(1)} + C^{(3),(3)}. \quad (46)$$

Following the monodromy group analysis of the roots λ_i in [48, 7, 8, 11], we find that $C^{(1),(1)}$ for **10** cannot be realized on the surface. Therefore there are only two **10** curves. The corresponding solutions for the matter curves in terms of λ_i after factorization are:

$$\begin{aligned} \Sigma_{\mathbf{16}}^{(1)} &: \quad \{\lambda_4\} \\ \Sigma_{\mathbf{16}}^{(3)} &: \quad \{\lambda_1, \lambda_2, \lambda_3\} \\ \Sigma_{\mathbf{10}}^{(13)} &: \quad \{\lambda_1 + \lambda_4, \lambda_2 + \lambda_4, \lambda_3 + \lambda_4\}, \\ \Sigma_{\mathbf{10}}^{(33)} &: \quad \{\lambda_1 + \lambda_2, \lambda_2 + \lambda_3, \lambda_3 + \lambda_1\}. \end{aligned} \quad (47)$$

	$C^{(1),(1)}$	$C^{(1),(3)} + C^{(3),(1)}$	$C^{(3),(3)}$
16	$\sigma \cdot \pi^* \xi$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi)$
10	$\pi^* \xi \cdot \pi^*(c_1 + \xi)$	$\frac{2(\sigma + \pi^*(c_1 + \xi))}{\cdot \pi^*(\eta - 3c_1 - \xi) + 2\sigma \cdot \pi^* \xi}$	$\frac{(2\sigma + \pi^*(\eta - 2c_1 - \xi))}{\cdot \pi^*(\eta - 3c_1 - \xi) + 2(\sigma + \pi^* c_1) \cdot \pi^* \xi}$
∞	$\sigma_c \cdot \pi^*(c_1 + \xi)$	$4\sigma_c \cdot \pi^*(c_1 + \xi)$	$\frac{\sigma_c \cdot \pi^*(\eta - c_1 - \xi)}{+ 2\sigma_c \cdot \pi^*(\eta - 2c_1 - 2\xi)}$

Table 4: Factorization $C_V = C^{(1)} + C^{(3)}$

The matter curves corresponding to $C^{(1)}$ and $C^{(3)}$ are $\Sigma_a^{(1)}$ and $\Sigma_b^{(3)}$, where:

$$\Sigma_a^{(1)} = C^{(1)} \cdot \sigma, \quad \Sigma_b^{(3)} = C^{(3)} \cdot \sigma. \quad (48)$$

The details about the components of the curves are listed in Table 4⁵. In what follows, we summarize some properties of this $(1, 3)$ cover factorization. Our notation follows [8, 11].

⁵In the table we abused notation. The covers for the two **16** curves should actually be denoted by $C^{(1)}$ and $C^{(3)}$.

3.4 Universal flux

For an $SU(n)$ cover the first Chern class vanishes: $c_1(V) = 0$. Since the cover factorizes into $C^{(1)}$ and $C^{(3)}$, by using $c_1(V_3 \oplus L) = c_1(V_3) + c_1(L)$, the traceless condition turns out to be:

$$c_1(p_{1*}\mathcal{L}^{(1)}) + c_1(p_{3*}\mathcal{L}^{(3)}) = 0, \quad (49)$$

where again $p_i : C^{(i)} \rightarrow S$. For the flux to be well defined and supersymmetric, there are two more conditions [11]:

$$\mathcal{L}^{(i)} \in H^2(C^{(i)}, \mathbb{Z}) \quad (50)$$

$$c_1(p_{1*}\mathcal{L}^{(1)}) - c_1(p_{3*}\mathcal{L}^{(3)}) \text{ is SUSY on } S. \quad (51)$$

More details about these conditions will be discussed below.

The ramification divisor for the map p_i on the cover C_V is $r = p^*c_1 - c_1(C_V)$. So for each component of the split cover we get:

$$r_i = p_i^*c_1 - c_1(C^i), \quad c_1(C^i) = (c_1(T_{\bar{X}}) - C^i) \cdot C^i. \quad (52)$$

The ramification divisors for a $(1, 3)$ factorization are:

$$r_a^{(1)} = (-\sigma + \pi^*\xi) \cdot C^{(1)}, \quad r_b^{(3)} = (\sigma + \pi^*(\eta - 2c_1 - \xi)) \cdot C^{(3)}. \quad (53)$$

Using the formula $c_1(p_{i*}\mathcal{L}^{(i)}) = p_{i*}c_1(\mathcal{L}^{(i)}) - \frac{1}{2}p_{i*}r_i$ we can then define the flux on each cover:

$$\gamma_i = c_1(\mathcal{L}^{(i)}) - \frac{1}{2}r_i. \quad (54)$$

3.4.1 Splitting of the universal flux

The universal flux encoded in the $n = 4$ spectral cover has been given in (35). This flux is also separated due to the split spectral cover. The corresponding universal flux on each factorized curve is:

$$\begin{aligned} \gamma_a &= \Sigma_a - p_a^*p_{a*}\Sigma_a = C^{(1)} \cdot (\sigma - \pi^*\xi), \\ \gamma_b &= 3\Sigma_b - p_b^*p_{b*}\Sigma_b = C^{(3)} \cdot (3\sigma - \pi^*(\eta - 4c_1 - \xi)). \end{aligned} \quad (55)$$

Motivated by the expansion of γ_u when splitting the cover, there are two more choices of fluxes:

$$\begin{aligned} \delta_a &= 3\Sigma_a - p_b^*p_{a*}\Sigma_a = C^{(1)} \cdot 3\sigma - C^{(3)} \cdot \pi^*\xi, \\ \delta_b &= \Sigma_b - p_a^*p_{b*}\Sigma_b = C^{(3)} \cdot \sigma - C^{(1)} \cdot \pi^*(\eta - 4c_1 - \xi). \end{aligned} \quad (56)$$

In addition, there is a third type of universal flux $\tilde{\rho}$ that can be included [11]:

$$\tilde{\rho} = 3\pi_a^*\rho - \pi_b^*\rho, \quad \rho \in H_2(S, \mathbb{R}). \quad (57)$$

ρ does not have to be effective because we can build $\tilde{\rho}$ with any real linear combination of $\tilde{\rho}_i$ from effective ρ_i [11].

We summarize the contributions of the above flux components to the chiralities of each factorized matter curve in the following two tables:

	class in S	γ_a	γ_b
$\mathbf{16}_a^{(1)}$	ξ	$-\xi(c_1 + \xi)$	0
$\mathbf{16}_b^{(3)}$	$\eta - 4c_1 - \xi$	0	$-(\eta - c_1 - \xi) \cdot (\eta - 4c_1 - \xi)$
$\mathbf{10}_{ab}^{(1,3)}$	$\eta - 3c_1$	$-\xi(c_1 + \xi)$	$-(\eta - 3c_1 - 3\xi) \cdot (\eta - 4c_1 - \xi)$
$\mathbf{10}_{bb}^{(3,3)}$	$\eta - 3c_1$	0	$(\eta - 3c_1 - 3\xi) \cdot (\eta - 4c_1 - \xi)$

(58)

	δ_a	δ_b	$\tilde{\rho}$
$\mathbf{16}_a^{(1)}$	$-3c_1\xi$	$-\xi \cdot (\eta - 4c_1 - \xi)$	$3\rho \cdot \xi$
$\mathbf{16}_b^{(3)}$	$-\xi \cdot (\eta - 4c_1 - \xi)$	$-c_1 \cdot (\eta - 4c_1 - \xi)$	$-\rho \cdot (\eta - 4c_1 - \xi)$
$\mathbf{10}_{ab}^{(1,3)}$	$\xi \cdot (2\eta - 9c_1 - 3\xi)$	$-(\eta - 3c_1 - \xi) \cdot (\eta - 4c_1 - \xi)$	$2 \cdot \rho(\eta - 3c_1)$
$\mathbf{10}_{bb}^{(3,3)}$	$-2\xi \cdot (\eta - 3c_1)$	$(\eta - 3c_1 - \xi) \cdot (\eta - 4c_1 - \xi)$	$-2\rho \cdot (\eta - 3c_1)$

(59)

The total universal flux is then the linear combination of these pieces:

$$\gamma_u = k_a \gamma_a + k_b \gamma_b + d_a \delta_a + d_b \delta_b + \tilde{\rho} \quad (60)$$

Thus, the fluxes on each component of the split cover are:

$$\begin{aligned} \gamma_{ua} &= C^{(1)} \cdot [(k_a + 3d_a)\sigma - \pi^*(k_a\xi + d_b(\eta - 4c_1 - \xi) - 3\rho)], \\ \gamma_{ub} &= C^{(3)} \cdot [(3k_b + d_b)\sigma - \pi^*(k_b(\eta - 4c_1 - \xi) + d_a\xi + \rho)], \end{aligned} \quad (61)$$

where

$$\begin{aligned} p_{a*}\gamma_{ua} &= 3d_a\xi - d_b(\eta - 4c_1 - \xi) + 3\rho \\ p_{b*}\gamma_{ub} &= -3d_a\xi + d_b(\eta - 4c_1 - \xi) - 3\rho \end{aligned} \quad (62)$$

The coefficients have to obey the following quantization conditions [11]:

$$\begin{aligned} (k_a + 3d_a + \frac{1}{2})\sigma - \pi^*(k_a\xi + d_b(\eta - 4c_1 - \xi) - 3\rho - \frac{1}{2}\xi) &\in H_4(\bar{X}, \mathbb{Z}), \\ (3k_b + d_b - \frac{1}{2})\sigma - \pi^*(k_b(\eta - 4c_1 - \xi) + d_a\xi + \rho - \frac{1}{2}(\eta - 2c_1 - \xi)) &\in H_4(\bar{X}, \mathbb{Z}). \end{aligned} \quad (63)$$

3.4.2 Chirality on the curves

From (60) we can summarize the number of generations on the matter curves as:

$$n_{\mathbf{16}_a^{(1)}} = (d_b - k_a)\xi^2 - d_b\xi\eta + (4d_b - k_a - 3d_a)\xi c_1 + 3\rho\xi, \quad (64)$$

$$\begin{aligned} n_{\mathbf{16}_b^{(3)}} &= -k_b(\eta^2 + 4c_1^2 - 5c_1\eta) + d_b(4c_1^2 - c_1\eta) + (d_a - k_b)\xi^2 + (2k_b - d_a)\xi\eta \\ &\quad + (4d_a - 5k_b + d_b)\xi c_1 - \rho\eta + 4\rho c_1 + \rho\xi, \end{aligned} \quad (65)$$

$$\begin{aligned} n_{\mathbf{10}_{ab}} &= -(k_b + d_b)(\eta^2 - 7c_1\eta + 12c_1^2) - (k_a + 3k_b + d_b + 3d_a)\xi^2 + (4k_b + 2d_b + 2d_a)\xi\eta \\ &\quad - (k_a + 15k_b + 7d_b + 9d_a)\xi c_1 + 2\rho\eta - 6\rho c_1, \end{aligned} \quad (66)$$

$$\begin{aligned} n_{\mathbf{10}_{bb}} &= (k_b + d_b)(\eta^2 - 7c_1\eta + 12c_1^2) + (k_b + d_b)\xi^2 - (4k_b + 2d_b + 2d_a)\xi\eta \\ &\quad + (15k_b + 7d_b + 6d_a)\xi c_1 - 2\rho\eta + 6\rho c_1, \end{aligned} \quad (67)$$

3.4.3 Tadpole condition

The tadpole condition for D3-branes is

$$N_{D3} = \frac{\chi(Y)}{24} - \frac{1}{2} \int_Y G \wedge G \quad (68)$$

If there are non-abelian singularities in Y , they account for additional contributions to the Euler characteristics, which are [10]:

$$\chi_{SU(n)} = \int_{S_{GUT}} [c_1^2(n^3 - n) + 3n\eta(\eta - nc_1)] \quad (69)$$

$$\chi_{E_8} = \int_{S_{GUT}} 120(-27c_1\eta + 62c_1^2 + 3\eta^2) . \quad (70)$$

For $n = 4$,

$$\chi(X_4) = \chi(X_4^*) + \chi_{SU(4)} - \chi_{E_8} \quad (71)$$

These equations were derived for models with a heterotic dual and conjectured to hold also for more general F-theory models in [10]. We will put this formula to the test for the examples we will present in Section 4. For three out of five examples find a discrepancy between the Euler numbers computed by (71) and those computed from the geometry of the Calabi-Yau fourfold. We will discuss possible causes for this discrepancy in the Conclusions.

For a $(1, 3)$ factorization the bundle changes as follows: $SU(4) \rightarrow S[U(3) \times U(1)]$. As a consequence the Euler number becomes [10, 11]:

$$\chi(X_4) = \chi(X_4^*) + \chi_{SU(1)} + \chi_{SU(3)} - \chi_{E_8} \quad (72)$$

The class η in (69) is decomposed into:

$$\eta^{(3)} = \eta - (c_1 + \xi), \quad \eta^{(1)} = c_1 + \xi. \quad (73)$$

Thus one gets:

$$\chi(X_4) = \chi(X_4^*) + \int_S 3(3\eta^2 + 20c_1^2 - 15c_1\eta + 4\xi^2 - 6\xi\eta + 16\xi c_1) - \chi_{E_8} \quad (74)$$

On the other hand we have:

$$\frac{1}{2} \int_{X_4} G \wedge G = -\frac{1}{2} \gamma^2 = -\frac{1}{2} (\gamma_a \cdot \gamma_a + \gamma_b \cdot \gamma_b), \quad (75)$$

and

$$\begin{aligned} \gamma^2 &= -(k_a + 3d_a)^2 \xi(c_1 + \xi) - \frac{1}{3} (3k_b + d_b)^2 (\eta - c_1 - \xi)(\eta - 4c_1 - \xi) \\ &\quad + \frac{4}{3} [d_b(\eta - 4c_1 - \xi) - 3d_a\xi - 3\rho]^2 \end{aligned} \quad (76)$$

The D3-branes are added to cancel the tadpoles. We want to avoid a situation with anti-D3-branes, therefore the constraint is:

$$N_{D3} \geq 0. \quad (77)$$

3.4.4 Supersymmetry condition for the universal flux

We have the condition that $c_1(p_{a*}\mathcal{L}_a) - c_1(p_{b*}\mathcal{L}_b)$ is the Poincaré dual of a supersymmetric cycle in S . Recall that \mathcal{L} is the line bundle on the cover and p_i denotes the projection map from $C^{(i)}$ to S . For the flux γ , the supersymmetry constraint restricted on S is:

$$\omega \cdot (p_{i*}\gamma_{ui}) = 0, \quad (78)$$

where $p_{i*}\gamma_{ui}$ is as in (62) in the $(1, 3)$ splitting. ω is an ample divisor dual to Kähler form J of S :

$$\omega = \alpha_i \mathbf{C}_i, \quad (79)$$

where the \mathbf{C}_i are the generators of the Mori cone of dP_n , and $\alpha_i > 0$.

4 Explicit examples

4.1 Geometric backgrounds

The GUT surface is a del Pezzo surface as well as a divisor in the base B . Since we are interested in the physics on the GUT surface, it is more convenient to extract the useful information from the ambient space. We will present the geometry of the del Pezzo surface dP_k in terms of the hyperplane divisor H of \mathbb{P}^2 and the exceptional divisors E_i , $i = 1, 2, \dots, k$ with intersection numbers

$$H \cdot H = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij}, \quad \forall i, j. \quad (80)$$

The canonical divisor of dP_k is then

$$K_S = -3H + \sum_i^k E_i. \quad (81)$$

In what follows we will discuss five examples.

4.1.1 Model 1

The starting point for this model is the Fano hypersurface $\mathbb{P}^4[4]$ where we perform two curve blowups. The weight matrix for the blowup of one curve is given in table 11 in Appendix A. The second curve blowup is realized by adding line two of table 12 to the weight matrix in table 11 whereby one adds a seventh column to table 11 filled with zeros. For convenience reasons we rearrange this a bit and obtain the following table:

y_1	y_2	y_3	y_4	y_5	y_6	y_7	\sum
1	1	0	0	1	1	1	5
1	0	0	1	0	0	1	3
0	1	1	0	0	0	1	3
			J_1	J_2		J_3	

(82)

In the tables in Appendix A, models with this toric ambient space are labeled by $2C0P2$. There are at least two possible triangulations for the toric space. The relevant one here has the label 1 in Appendix A. The hypersurface describing the base manifolds has degrees $(4, 2, 2)$.

The GUT divisor S ($y_4 = 0$) is dP_5 . To see this, we look at the submanifold of the toric ambient space after setting y_4 to zero. Since by the Stanley-Reisner ideal of the fourfold, y_4 and y_1 are not allowed to vanish simultaneously, we obtain the following equivalence relations:

$$(1, y_2, y_3, 0, y_5, y_6, \frac{y_7}{y_1}) \sim (1, \rho\lambda y_2, \lambda y_3, 0, \rho y_5, \rho y_6, \lambda \frac{y_7}{y_1}) \quad \forall \rho, \lambda \in \mathbb{C}^*, \quad (83)$$

where y_2, \dots, y_6 and $\frac{y_7}{y_1}$ are now the new homogeneous coordinates. Via a Segre like map we can embed this threefold \mathcal{G} into \mathbb{P}^4 , $y \mapsto [y_2 : y_3 y_5 : \frac{y_7}{y_1} y_6 : y_3 y_6 : \frac{y_7}{y_1} y_5] \in \mathbb{P}^4$. The Stanley-Reisner ideal guarantees that we do not map to $\vec{0}$. Since the points of \mathcal{G} fulfill a certain relation in terms of the homogeneous coordinates of \mathbb{P}^4 , \mathcal{G} is realized as a hypersurface of degree two in \mathbb{P}^4 . From the hypersurface equation, which is compatible with this map, one obtains a second degree two equation. Thus, we have a complete intersection of two degree two equations in \mathbb{P}^4 , which is in fact a dP_5 .

The relevant first Chern classes of the base manifold in terms of the ambient space divisor basis $\{J_1, J_2, J_3\}$ are:

Chern class	in B	on S
$c_1(B)$	J_3	-
$c_1(N_{S B})$	J_1	$AH + \sum_{i=1}^5 B_i E_i$
$c_1(S)$	$J_3 - J_1$	$3H - \sum_{i=1}^5 E_i$

(84)

where A and B_i are integers and will be determined. The triple intersections are:

$$J_1^2 J_3 = -2, \quad J_2 J_3^2 = 4, \quad J_1 J_2 J_3 = 2, \quad J_3^3 = 4. \quad (85)$$

Using equations (9), we calculate the volumes of the base B and the GUT brane S in terms of the Kähler moduli $r_i > 0$:

$$\begin{aligned} \text{Vol}(B) &= 16r_1^3 + 36r_1^2 r_2 + 24r_1 r_2^2 + 4r_2^3 + 24r_1^2 r_3 + 36r_1 r_2 r_3 + 12r_2^2 r_3 + 6r_1 r_3^2 + 6r_2 r_3^2 \\ \text{Vol}(S) &= 4r_1^2 + 4r_1 r_2 \end{aligned} \quad (86)$$

Thus, for $r_3 \rightarrow \infty$ the GUT divisor remains of finite size while the volume of the base becomes infinitely large. One can also check that in this limit all the other divisors inside B also obtain infinite volume. For $r_1 = 0$ and at least $r_2 \neq 0$ we can implement the mathematical decoupling limit. For $r_1 = r_3 = 0$ a further divisor which is not del Pezzo will shrink to zero size.

The geometry for both pictures in (84) should be consistent. We can use the triple intersections in (85) to compute $c_1(N_{S|B})^2 = J_1^3 = 0$ and $c_1(N_{S|B}) \cdot c_1(S) = J_1^2(J_3 - J_1) = -2$ on S . There are two possible solutions for A and the B_i 's, which are

$$\begin{aligned} c_1(N_{S|B}) &= -H + E_i; \\ \text{or } &-2H + E_i + E_j + E_k + E_l, \quad i \neq j \neq k \neq l. \end{aligned} \quad (87)$$

These two solutions are actually equivalent via an involution ($I_{B_5}^{(5)}$) of the del Pezzo surface. Its explicit action can be found in Table 22 of [49]. For concreteness we will stick to $c_1(N_{S|B}) = -H + E_5$ in the following.

Finally we can collect the information we need for model building,

$$\begin{array}{llll}
c_1(S) & c_1 & 3H - E_1 - E_2 - E_3 - E_4 - E_5 \\
c_1(N_{S|B}) & -t & -H + E_5 \\
b_0 = \eta & 6c_1 - t & 17H - 6(E_1 + E_2 + E_3 + E_4 + E_5) + E_5 \\
b_2 & \eta - 2c_1 & 11H - 4(E_1 + E_2 + E_3 + E_4 + E_5) + E_5 \\
b_3 & \eta - 3c_1 & 8H - 3(E_1 + E_2 + E_3 + E_4 + E_5) + E_5 \\
b_4 & \eta - 4c_1 & 5H - 2(E_1 + E_2 + E_3 + E_4 + E_5) + E_5
\end{array} \quad (88)$$

4.1.2 Model 2

Our second model is a point blowup in $\mathbb{P}^4[4]$. The corresponding weight matrix can be found in table 19 in Appendix A. We label the model by 0C1P1. The base B is a hypersurface of degree $(4, 1)$. The relevant first Chern classes of the base in terms of the ambient space divisor basis $\{J_1 \sim [y_2], J_2 \sim [y_1]\}$ are:

Chern class	in B	on S
$c_1(B)$	J_2	-
$c_1(N_{S B})$	$J_2 - J_1$	$AH + \sum_i^6 B_i E_i$
$c_1(S)$	J_1	$3H - \sum_i^6 E_i$

(89)

where A and B_i are integers and will be determined. The triple intersections are:

$$J_1^3 = 1, \quad J_1^2 J_2 = 4, \quad J_1 J_2^2 = 4, \quad J_2^3 = 4. \quad (90)$$

We can check the existence of a decoupling limit by computing the volumes of the base B and the GUT divisor S in terms of $r_i > 0$:

$$\begin{aligned}
\text{Vol}(B) &= 4r_1^3 + 12r_1^2 r_2 + 12r_1 r_2^2 + r_2^3 \\
\text{Vol}(S) &= 3r_2^2
\end{aligned} \quad (91)$$

For $r_1 \rightarrow \infty$ we find the physical decoupling limit. In this limit no other divisor in B remains of finite size. Setting $r_2 = 0$ while keeping $r_1 \neq 0$ gives the mathematical decoupling limit. No divisors other than the GUT divisor will shrink to zero size in this limit.

The geometry for both pictures in (89) should be consistent. We calculate $c_1(N_{S|B})^2 = 3$ and $c_1(N_{S|B}) \cdot c_1(S) = -3$ on S . Therefore one finds there is only one possible solution in this case:

$$c_1(N_{S|B}) = -3H + E_1 + E_2 + E_3 + E_4 + E_5 + E_6. \quad (92)$$

Finally we can write down the information we need for model building⁶:

$$\begin{array}{llll}
c_1(S) & c_1 & 3H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 \\
c_1(N_{S|B}) & -t & -3H + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 \\
b_0 = \eta & 6c_1 - t & 15H - 5(E_1 + E_2 + E_3 + E_4 + E_5 + E_6) \\
b_2 & \eta - 2c_1 & 9H - 3(E_1 + E_2 + E_3 + E_4 + E_5 + E_6) \\
b_3 & \eta - 3c_1 & 6H - 2(E_1 + E_2 + E_3 + E_4 + E_5 + E_6) \\
b_4 & \eta - 4c_1 & 3H - (E_1 + E_2 + E_3 + E_4 + E_5 + E_6)
\end{array} \quad (93)$$

⁶Note that one can only construct an $SO(10)$ model in this geometry. In an $SU(5)$ model one would not obtain a **10** curve.

4.1.3 Model 3

The third model we would like to discuss is a blowup of one curve and one point in the Fano $\mathbb{P}^4[4]$. The weight matrix is written in table 14 in Appendix A where one has to replace the first '*' by a 0 and the second '*' by a 1. We give the model the identifier 1C1P2. The base manifold is a hypersurface of degree $(4, 2, 1)$. The GUT divisor S ($y_6 = 0$) is dP_4 . To make this explicit we again set the coordinate corresponding to the GUT divisor to zero and fix one scaling, in this case the second one ($y_5 = 1$). What we obtain from the new equivalence relations and the Stanley-Reisner ideal is $\mathbb{P}^2 \times \mathbb{P}^1$. The hypersurface becomes a degree $(2, 1)$ equation in the new degrees. Thus, we can bring it to the following form:

$$y_4 g_1(y_1, y_2, y_3) = y_7 g_2(y_1, y_2, y_3). \quad (94)$$

where $[y_4 : y_7]$ and $[y_1 : y_2 : y_3]$ are the homogeneous coordinates of \mathbb{P}^1 and \mathbb{P}^2 , respectively. Hence, the four points in \mathbb{P}^2 where $g_1 = g_2 = 0$ vanish are blown up to four \mathbb{P}^1 s, i.e. S is a dP_4 .

The relevant first Chern classes of the base in terms of the ambient space divisor basis $\{J_1 \sim [y_3], J_2 \sim [y_4], J_3 \sim [y_5]\}$ are:

Chern class	in B	on S
$c_1(B)$	J_2	-
$c_1(N_{S B})$	$J_3 - J_1$	$AH + \sum_{i=1}^4 B_i E_i$
$c_1(S)$	$J_1 + J_2 - J_3$	$3H - \sum_{i=1}^4 E_i$

(95)

where A and B_i are integers that will be determined below. The triple intersections are:

$$\begin{aligned} J_1^2 J_2 = 2, \quad J_1^2 J_3 = 1, \quad J_2^3 = 4, \quad J_1 J_2^2 = 4, \quad J_2^2 J_3 = 4, \\ J_3^3 = 1, \quad J_1 J_3^2 = 1, \quad J_2 J_3^2 = 4, \quad J_1 J_2 J_3 = 4. \end{aligned} \quad (96)$$

The volumes of the base B and the GUT divisor S can be determined from (9):

$$\begin{aligned} \text{Vol}(B) &= 4r_1^3 + 12r_1^2 r_2 + 12r_1 r_2^2 + r_2^3 + 12r_1^2 r_3 + 24r_1 r_2 r_3 + 3r_2^2 r_3 + 6r_1 r_3^2 + 3r_2 r_3^2 \\ \text{Vol}(S) &= 4r_1 r_3 + r_3^2 \end{aligned} \quad (97)$$

The limit $r_2 \rightarrow \infty$ keeps the volume of the GUT divisor finite while sending the volumes of the base and all other divisors to infinity. To obtain a mathematical decoupling limit we have to set $r_3 = 0$ while keeping r_1 or r_2 non-zero. Note that if we also set $r_2 = 0$ the volume of the base will still be finite but another divisor which does not have the topological characteristics of a del Pezzo will shrink to zero size.

The geometry for both pictures in (95) should be consistent. We compute $c_1(N_{S|B})^2 = 1$ and $c_1(N_{S|B}) \cdot c_1(S) = -3$ on S . Then one finds two solutions for A and the B_i 's:

$$\begin{aligned} c_1(N_{S|B}) &= -2H + E_i + E_j + E_k, \quad i \neq j \neq k; \\ \text{or} \quad &-H. \end{aligned} \quad (98)$$

These solutions are related by automorphisms of the del Pezzo surface⁷. Thus, the solutions are equivalent modulo automorphisms. For concreteness we will only use the last solution of (98) in the following.

Finally we can collect the information we need for model building:

$$\begin{array}{lll}
c_1(S) & c_1 & 3H - E_1 - E_2 - E_3 - E_4 \\
c_1(N_{S|B}) & -t & -H \\
b_0 = \eta & 6c_1 - t & 17H - 6(E_1 + E_2 + E_3 + E_4) \\
b_2 & \eta - 2c_1 & 11H - 4(E_1 + E_2 + E_3 + E_4) \\
b_3 & \eta - 3c_1 & 8H - 3(E_1 + E_2 + E_3 + E_4) \\
b_4 & \eta - 4c_1 & 5H - 2(E_1 + E_2 + E_3 + E_4)
\end{array} \tag{100}$$

4.1.4 Model 4

We consider a blowup of two curves and one point starting from $\mathbb{P}^4[4]$. The weight matrix for this model can be found in table 18 in Appendix A, where one should replace the third '*' by 1 and all the others by 0. In our data tables this is labeled by *2C1P4*. The hypersurface we consider in this ambient space has degrees $(4, 2, 2, 1)$. The GUT divisor S ($y_7 = 0$) is a dP_5 ⁸. To show that we set y_7 to zero and 'scale away' the third row of table 18. We are allowed to do this because by the Stanley-Reisner ideal of this triangulation y_7 and y_2 must not vanish at the same time. The remaining weight matrix looks as follows:

y_1	y_3	$\frac{y_4}{y_7}$	y_5	y_6	y_8
1	1	0	1	0	0
0	0	1	1	1	0
0	0	1	0	0	1

$$\text{with SR-I} = \left\{ \frac{y_4}{y_7} y_8, y_5 y_6, y_1 y_3 y_5, y_8 y_1 y_3 \right\}. \tag{101}$$

On this submanifold the hypersurface equation takes the form,

$$y_8 g_1^{(2,2,0)} = \frac{y_4}{y_7} g_2^{(2,1,0)}, \tag{102}$$

where $g_1^{(2,2,0)}$ and $g_2^{(2,1,0)}$ are homogeneous functions of the indicated degree. Looking at the equivalence relations of the homogeneous coordinates appearing in g_1 and g_2 we observe that they are the ones of a dP_1 . Also the scalings are correct, y_1 and y_3 are allowed to vanish simultaneously. However, $y_1 = y_3 = 0$ is not a solution to the hypersurface equation so we can safely exclude it from the definition set. Equation (102) tells us that the points of dP_1 where $g_1 = g_2 = 0$ vanish are replaced by further point blowups (\mathbb{P}^1). Thus, we end up with a dP_5 .

⁷To convince oneself that this is the case, one can have a look at the (dual) intersection graph of (-1)-curves of dP_4 , see the left graph of Figure 9 in [49] (Petersen graph). The graph has an obvious \mathbb{Z}_5 symmetry. If one assigns $H - E_4 - E_2$, $H - E_3 - E_4$, $H - E_1 - E_3$, E_4 and E_3 to the inner points and performs a positive rotation one obtains the following transformations:

$$H \mapsto 2H - E_1 - E_2 - E_4 \mapsto 2H - E_2 - E_3 - E_4 \mapsto 2H - E_1 - E_3 - E_4 \mapsto 2H - E_1 - E_2 - E_3 \tag{99}$$

⁸Note that there is a second dP_5 in this geometry which also satisfies both the mathematical and the physical decoupling limit. However, there occurs a problem with the fourfold for this divisor: After removing points in the M-lattice in order to impose the $SO(10)$ gauge group the polyhedron describing the fourfold is no longer reflexive.

The relevant first Chern classes of the base in terms of the ambient space divisor basis $\{J_1 \sim [y_1], J_2 \sim [y_2], J_3 \sim [y_4], J_4 \sim [y_5]\}$ are:

Chern class	in B	on S
$c_1(B)$	J_3	-
$c_1(N_{S B})$	$J_2 - J_1$	$AH + \sum_{i=1}^5 B_i E_i$
$c_1(S)$	$J_1 - J_2 + J_3$	$3H - \sum_{i=1}^5 E_i$

(103)

where A and B_i are integers and will be determined. The triple intersections are:

$$\begin{aligned}
J_2^2 J_3 &= 2, & J_2^2 J_4 &= 1, & J_1 J_3^2 &= 4, & J_2 J_3^2 &= 4, & J_3^2 J_4 &= 4, & J_3^3 &= 4, & J_2 J_4^2 &= 1, \\
J_3 J_4^2 &= 2, & J_1 J_2 J_3 &= 2, & J_1 J_2 J_4 &= 1, & J_1 J_3 J_4 &= 2, & J_2 J_3 J_4 &= 4.
\end{aligned}
\tag{104}$$

In order to study the decoupling limit we calculate the volumes of the base and the GUT brane in terms of $r_i > 0$ using (9):

$$\begin{aligned}
\text{Vol}(B) &= 3r_1^2 r_2 + 3r_1 r_2^2 + 6r_1^2 r_3 + 24r_1 r_2 r_3 + 6r_2^2 r_3 + 12r_1 r_3^2 + 12r_2 r_3^2 + 4r_3^3 + 6r_1^2 r_4 + 30r_1 r_2 r_4 \\
&\quad + 6r_2^2 r_4 + 36r_1 r_3 r_4 + 36r_2 r_3 r_4 + 24r_3^2 r_4 + 24r_1 r_4^2 + 24r_2 r_4^2 + 36r_3 r_4^2 + 16r_4^3 \\
\text{Vol}(S) &= r_1^2 + 4r_1 r_3 + 6r_1 r_4 + 4r_3 r_4 + 4r_4^2
\end{aligned}
\tag{105}$$

For $r_2 \rightarrow \infty$ the volume of the base becomes infinitely large while the volume of the GUT divisor remains finite. In this limit, also the volumes of all the other divisors become infinite. If we want to shrink the volume of the GUT divisor to zero size we have to set at least $r_1 = r_4 = 0$ while keeping r_3 finite. If we also set $r_2 = 0$ $\text{Vol}(B)$ will still be non-zero but two more dP_5 -divisors will have zero volume.

The geometry for both pictures in (103) should be consistent. We calculate $c_1(N_{S|B})^2 = 0$ and $c_1(N_{S|B}) \cdot c_1(S) = -2$ on S . Then one finds two possible solutions in this case:

$$\begin{aligned}
c_1(N_{S|B}) &= -H + E_i; \\
&\text{or } -2H + E_i + E_j + E_k + E_l, \quad i \neq j \neq k \neq l.
\end{aligned}$$

Again both solutions are related via an involution. To fix a basis we choose

$$c_1(N_{S|B}) = -H + E_5 \tag{106}$$

for the first Chern class of the normal bundle.

Finally we can write down the information we need for model building. Since it is also a dP_5 space and the conditions are the same as those in Model 1 on S , we again get:

$c_1(S)$	c_1	$3H - E_1 - E_2 - E_3 - E_4 - E_5$
$c_1(N_{S B})$	$-t$	$-H + E_5$
$b_0 = \eta$	$6c_1 - t$	$17H - 6(E_1 + E_2 + E_3 + E_4 + E_5) + E_5$
b_2	$\eta - 2c_1$	$11H - 4(E_1 + E_2 + E_3 + E_4 + E_5) + E_5$
b_3	$\eta - 3c_1$	$8H - 3(E_1 + E_2 + E_3 + E_4 + E_5) + E_5$
b_4	$\eta - 4c_1$	$5H - 2(E_1 + E_2 + E_3 + E_4 + E_5) + E_5$

(107)

There is no difference to (88).

4.1.5 Model 5

Our final example starts with the Fano $\mathbb{P}^4[3]$, and we blow up two curves and one point. The weight matrix can be found in Appendix A in table 17, where the '*' is to be replaced by 1 and the '◊' by 0. In the tables in Appendix A this model is labeled by 2C1P1. The hypersurface has degrees (3, 2, 1, 1). Due to singularities in the ambient space there are two triangulations. The particular triangulation we choose is denoted by 1 in the tables in Appendix A. The GUT divisor S ($y_7 = 0$) is dP_4 . To make this obvious, we use the same steps as in model 3 except that in this case the Stanley-Reisner ideal allows us to 'scale away' two rows, the second and the third. We end up again with $\mathbb{P}^2 \times \mathbb{P}^1$. The modified hypersurface equation, we obtain an equivalent version of model 3,

$$y_8 g_1^{(2)}\left(\frac{y_5}{y_6}, y_3, \frac{y_4}{y_6}\right) = \frac{y_1}{y_2} g_2^{(2)}\left(\frac{y_5}{y_6}, y_3, \frac{y_4}{y_6}\right), \quad (108)$$

where $[\frac{y_5}{y_6} : y_3 : \frac{y_4}{y_6}]$ and $[y_8 : \frac{y_1}{y_2}]$ are the homogeneous coordinates of \mathbb{P}^2 and \mathbb{P}^1 , respectively. Thus, S is a dP_4 .

The relevant first Chern classes of the base in terms of the ambient space divisor basis $\{J_1 \sim [y_5], J_2 \sim [y_1], J_3 \sim [y_2], J_4 \sim [y_6]\}$ are:

Chern class	in B	on S
$c_1(B)$	$J_2 + J_3 + J_4$	-
$c_1(N_{S B})$	$J_3 + J_4 - J_1$	$AH + \sum_{i=1}^4 B_i E_i$
$c_1(S)$	$J_1 + J_2$	$3H - \sum_{i=1}^4 E_i$

(109)

where A and B_i are integers and will be determined. The triple intersections are:

$$\begin{aligned} J_1^2 J_2 = 1, \quad J_1^2 J_3 = 1, \quad J_1 J_2^2 = 2, \quad J_2^2 J_4 = 2, \quad J_3^2 J_4 = 2, \quad J_3^3 = -2, \quad J_1 J_4^2 = -1, \\ J_2 J_4^2 = -1, \quad J_3 J_4^2 = -1, \quad J_1 J_2 J_3 = 2, \quad J_1 J_2 J_4 = 1, \quad J_1 J_3 J_4 = 1, \quad J_2 J_3 J_4 = 2. \end{aligned} \quad (110)$$

The volumes of the base and the GUT divisor can be computed using (9):

$$\begin{aligned} \text{Vol}(B) &= 6r_1^2 r_2 + 3r_1 r_2^2 + 6r_1^2 r_3 + 18r_1 r_2 r_3 + 3r_2^2 r_3 + 9r_1 r_3^2 + 3r_2 r_3^2 + r_3^3 + 6r_1^2 r_4 + 18r_1 r_2 r_4 \\ &\quad + 3r_2^2 r_4 + 18r_1 r_3 r_4 + 18r_2 r_3 r_4 + 9r_3^2 r_4 + 9r_1 r_4^2 + 9r_2 r_4^2 + 9r_3 r_4^2 + 3r_4^3 \\ \text{Vol}(S) &= 4r_1 r_2 + r_2^2 + 4r_2 r_4 \end{aligned} \quad (111)$$

The physical decoupling limit exists for $r_3 \rightarrow \infty$. In this limit also the volumes of the other divisors in the base become infinitely large. In order to implement a mathematical decoupling limit we have to set at least $r_2 = 0$ while keeping finite values for r_3 or r_4 . Setting also $r_3 = 0$ will make another non del Pezzo divisor shrink to zero size. For $r_1 = 0$ an extra dP_1 will get zero volume.

The geometry for both pictures in (109) should be consistent, and we compute $c_1(N_{S|B})^2 = 1$ and $c_1(N_{S|B}) \cdot c_1(S) = -3$ on S . One finds that there are two possible solutions in this case:

$$\begin{aligned} c_1(N_{S|B}) &= -2H + E_i + E_j + E_k, \quad i \neq j \neq k. \\ \text{or} \quad &-H, \end{aligned} \quad (112)$$

which are again related by the automorphisms of model 3. For concreteness we chose the last solution in (112).

Finally we can collect the information we need for model building. Since it is also a dP_4 space and the conditions are the same as those in Model 3 on S , we can again write:

$$\begin{array}{llll}
c_1(S) & c_1 & 3H - E_1 - E_2 - E_3 - E_4 \\
c_1(N_{S|B}) & -t & -H \\
b_0 = \eta & 6c_1 - t & 17H - 6(E_1 + E_2 + E_3 + E_4) \\
b_2 & \eta - 2c_1 & 11H - 4(E_1 + E_2 + E_3 + E_4) \\
b_3 & \eta - 3c_1 & 8H - 3(E_1 + E_2 + E_3 + E_4) \\
b_4 & \eta - 4c_1 & 5H - 2(E_1 + E_2 + E_3 + E_4)
\end{array} \tag{113}$$

There is no difference to (100).

4.2 Fourfolds

Here we give the explicit data of the Calabi-Yau fourfold(s) constructed from the base manifold of the first Model. The Calabi-Yau fourfolds corresponding to the other base manifolds can be found in Appendix B. For convenience we relabeled the vertices obtained from the base. The vertex corresponding to the GUT divisor is given by ν_4 and we associate to it the coordinate w . The additional vertices/coordinates obtained after dualizing the reduced M-lattice polytope are denoted with a tilde. Furthermore we compute the Euler numbers for the $SO(10)$ model and compare with the formula (71).

The vertices in the N-lattice are:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0)$	2 2 2 2	x
	$\nu_2 = (-2 \quad 0 \quad -1 \quad -1 \quad -1 \quad 0)$	3 3 3 3	y
	$\nu_3 = (0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0)$	0 0 0 1	z
	$\nu_4 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$	1 0 0 0	w
	$\nu_5 = (0 \quad -1 \quad 0 \quad -1 \quad 0 \quad 1)$	0 1 0 0	y_1
	$\nu_6 = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)$	0 0 1 0	y_2
∇_2	$\nu_7 = (0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -1)$	1 1 0 0	y_3
	$\nu_8 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	0 1 0 0	y_4
	$\nu_9 = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$	1 1 1 0	y_5
	$\nu_{10} = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	0 1 1 0	y_6

After reducing the M-lattice polytope to the $SO(10)$ case, we obtain for the dual N-lattice

polytope:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0)$	1 2 2 0 2 2	x
	$\nu_2 = (-2 \quad 0 \quad -1 \quad -1 \quad -1 \quad 0)$	2 3 3 0 3 3	y
	$\nu_3 = (0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0)$	0 0 0 0 0 1	z
	$\nu_5 = (0 \quad -1 \quad 0 \quad -1 \quad 0 \quad 1)$	0 1 0 0 0 0	y_1
	$\nu_6 = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)$	0 0 0 0 1 0	y_2
∇_2	$\nu_7 = (0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -1)$	2 1 2 1 0 0	y_3
	$\nu_8 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	0 1 0 0 0 0	y_4
	$\nu_9 = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$	2 1 2 1 1 0	y_5
	$\nu_{10} = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	0 1 0 0 1 0	y_6
	$\tilde{\nu}_{11} = (1 \quad 0 \quad -1 \quad -1 \quad -1 \quad 2)$	1 0 0 0 0 0	\tilde{y}_7
	$\tilde{\nu}_{12} = (0 \quad 0 \quad -1 \quad -1 \quad -1 \quad 1)$	0 0 0 1 0 0	\tilde{y}_8
	$\tilde{\nu}_{13} = (0 \quad 0 \quad -1 \quad -1 \quad -1 \quad 2)$	0 0 1 0 0 0	\tilde{y}_9

(115)

Note that the GUT divisor $\{w = 0\}$ no longer corresponds to a vertex after this procedure. However it is still a point in the polytope ∇_2 . Furthermore the additional vertices appear in ∇_2 and not in ∇_1 . The Euler number is 912. Here we find a discrepancy with the Euler number computed via (71) where the result is 672.

We also find a mismatch for the Euler numbers computed in these two ways for the models 3 and 4. For the models 2 and 5 they agree.

4.3 GUT models

In this subsection we will use the geometric backgrounds discussed in Section 4.1 to create examples of $SO(10)$ models with a split spectral cover. We demonstrate numerical results for each dP_n on S .

4.3.1 Examples based on Model 1, $S = dP_5$

From the normal bundle $c_1(N_{S|B}) = -H + E_5$, we have:

$$\eta = 17H - 6E_1 - 6E_2 - 6E_3 - 6E_4 - 5E_5. \quad (116)$$

Model 1A

It is natural to set $\xi = \mathcal{O}$. For this case only the $\mathbf{16}_b^{(3)}$ curve contributes to the Yukawa coupling, and there is no contribution from the matter curve associated to $C^{(1)}$. By assuming $\rho = xH - \sum y_i E_i$ it is possible to obtain a general spectrum. Thus by (58) and (59) the

contributions from the components of the universal flux to the curve chirality are:

curve	$k_a\gamma_a$	$k_b\gamma_b$	$d_a\delta_a$	$d_b\delta_b$	ρ	chirality
16_a	0	0	0	0	0	0
16_b	0	$-26k_b$	0	$-6d_b$	$-5x + 2\sum_{i=1}^4 y_i + y_5$	$-(26k_b + 6d_b) - 5x + 2\sum_{i=1}^4 y_i + y_5$
10_{ab}	0	$-14k_b$	0	$-14d_b$	$2(8x - 3\sum_{i=1}^4 y_i - 2y_5)$	$-14(k_b + k_d) + 2(8x - 3\sum_{i=1}^4 y_i - 2y_5)$
10_{bb}	0	$14k_b$	0	$14d_b$	$-2(8x - 3\sum_{i=1}^4 y_i - 2y_5)$	$14(k_b + k_d) - 2(8x - 3\sum_{i=1}^4 y_i - 2y_5)$

(117)

We have sufficiently many degrees of freedom from the dP_5 surface that we can tune the parameters to create a three generation $SO(10)$ model with the supersymmetry and tadpole conditions satisfied. The spectrum of the model can be summarized as follows:

curve	class	generation
16_a	\mathcal{O}	0
16_b	$5H - 2(E_1 + E_2 + E_3 + E_4) - E_5$	3
10_{ab}	$8H - 3(E_1 + E_2 + E_3 + E_4) - 2E_5$	k
10_{bb}	$8H - 3(E_1 + E_2 + E_3 + E_4) - 2E_5$	$-k$

(118)

In the table above we have indicated that the parameters can be tuned such that we get three generations of fermions and k Higgs fields. The **10** fields from the two different curves are conjugate and have the same generation number. This implies that the $SO(10)$ GUT spectrum has exotic **10** fields. In what follows we will present other examples with $\xi \neq \mathcal{O}$.

Model 1B

The structure for the models with $\xi \neq \mathcal{O}$ is plentiful. We present an example of an $SO(10)$ model in Table 5⁹:

k_a	k_b	d_a	d_b	ξ	ρ
-0.5	0.5	0	-1	$H - E_1 + E_4 + 2E_5$	$-H + E_4 + E_5$

Table 5: Parameters of Model 1B.

The matter spectrum and the corresponding classes are:

Matter	class with fixed ξ	generation
16_a	$H - E_1 + E_4 + 2E_5$	0
16_b	$4H - E_1 - 2E_2 - 2E_3 - 3E_4 - 3E_5$	3
10_{ab}	$8H - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 2E_5$	1
10_{bb}	$8H - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 2E_5$	-1

(119)

In this model the tadpole condition and (72) imply that $N_{D3} = 8$. The supersymmetry condition is not very constrained, and for simplicity we choose for ω in (79) the following

⁹In the following examples we impose the condition $(c_1 + \xi) \cdot \xi = 0$ in order to keep the ramification of the cover $C^{(1)}$ trivial.

special form (for dP_5):

$$\omega = \beta \left((2H - \sum_{i=1}^5 E_i) + \sum_{j \neq k}^5 (H - E_j - E_k) \right) + \alpha \sum_{l=1}^5 E_l. \quad (120)$$

The condition for ω being ample can be summarized as $5\alpha > \beta > 0$ and $5\beta > \alpha$. Then in this model we find $\alpha/\beta = 13/3$.

4.3.2 Example based on Model 2

In the previous section we obtained $c_1(N_{S|B}) = -c_1(S)$, and therefore $\eta = 5c_1(S)$, so that the coefficients of the exceptional divisors E_i of dP_6 in η are the same. This implies that we need to choose a non-trivial ξ if we want to reserve the freedom of having restriction to the **16** curves by the flux $F_X = E_i - E_j$. Here we give an example of a non-trivial ξ and a three-generation **16** curve:

k_a	k_b	d_a	d_b	ξ	ρ
-0.5	-1.5	0	-1	$2H - 2E_1 - E_2 - E_3$	$-H + E_1 + E_2 + E_3$

Table 6: Parameters of Model 2

The matter spectrum and the corresponding classes are:

Matter	class with fixed ξ	generation
$\mathbf{16}_a^{(1)}$	$2H - 2E_1 - E_2 - E_3$	0
$\mathbf{16}_b^{(3)}$	$H + E_1 - E_4 - E_5 - E_6$	3
$\mathbf{10}_{ab}$	$6H - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6$	3
$\mathbf{10}_{bb}$	$6H - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6$	-3

(121)

In this model we get $N_{D3} = 27$ from the tadpole condition, using (72) to compute the Euler number. We choose for the supersymmetry condition (79) the following special case:

$$\begin{aligned} \omega &= \beta \left(\sum_{m \neq n \neq p \neq q \neq r}^6 (2H - E_m - E_n - E_p - E_q - E_r) + \sum_{j \neq k}^6 (H - E_j - E_k) \right) + \alpha \sum_{l=1}^6 E_l \\ &= 27\beta H - (10\beta - \alpha) \sum_{i=1}^6 E_i \end{aligned} \quad (122)$$

Then we get $10\beta > \alpha > 0$ and $\alpha/\beta = 16/35$.

4.3.3 Examples based on Model 5

On this dP_4 surface we choose the first solution for $c_1(N_{S|B})$ in (112). With that we obtain:

$$\eta = 17H - 6E_1 - 6E_2 - 6E_3 - 6E_4. \quad (123)$$

Again in this model it is possible to choose $\xi = \mathcal{O}$, analogous to what we discussed in Model 1. Here we present two examples with ξ non-trivial.

Example 5A

In this example we present a three-generation model. The parameters in this example are:

k_a	k_b	d_a	d_b	ξ	ρ
-1.5	-1.5	0	0	$H - E_1 - E_2 + E_3 + E_4$	$-H + 2E_1 + 2E_2 + E_3 + 2E_4$

Table 7: Parameters of Example 5A.

The matter spectrum and the corresponding classes are:

Matter	class with fixed ξ	generation
$\mathbf{16}_a^{(1)}$	$H - E_1 - E_2 + E_3 + E_4$	0
$\mathbf{16}_b^{(3)}$	$4H - E_1 - E_2 - 3E_3 - 3E_4$	3
$\mathbf{10}_{ab}$	$8H - 3E_1 - 3E_2 - 3E_3 - 3E_4$	2
$\mathbf{10}_{bb}$	$8H - 3E_1 - 3E_2 - 3E_3 - 3E_4$	-2

(124)

In this model the tadpole cancellation condition and (72) imply that $N_{D3} = 81$. For simplicity the supersymmetry condition in (79) is chosen to have the following form:

$$\omega = \beta \left(\sum_{j \neq k}^4 (H - E_j - E_k) \right) + \alpha \sum_{l=1}^4 E_l, \quad (125)$$

where $3\beta > \alpha > 0$. For this model we find $\alpha/\beta = 15/7$.

Example 5B

In this example we present a four-generation model in $SO(10)$. The reason to consider this situation is that if the flux F_X has non-zero restriction on the $\mathbf{16}$ curve, the chirality of the matter representations from this $\mathbf{16}$ curve will be modified such that the model may no longer have a three-generation interpretation in the flipped $SU(5)$ picture. The parameters in this example are listed in Table 8.

k_a	k_b	d_a	d_b	ξ	ρ
-2	-1	-0.5	1.5	$H - E_1 - E_2 + E_4$	$\frac{1}{2}H + E_2 + 2E_3$

Table 8: Parameters of Example 5B.

The matter content and the corresponding classes are:

Matter	class with fixed ξ	generation
$\mathbf{16}_a^{(1)}$	$H - E_1 - E_2 + E_4$	0
$\mathbf{16}_b^{(3)}$	$4H - E_1 - E_2 - 2E_3 - 3E_4$	4
$\mathbf{10}_{ab}$	$8H - 3E_1 - 3E_2 - 3E_3 - 3E_4$	8
$\mathbf{10}_{bb}$	$8H - 3E_1 - 3E_2 - 3E_3 - 3E_4$	-8

(126)

In this model the tadpole cancellation condition and (72) imply that $N_{D3} = 111$. The supersymmetry condition implies $\alpha/\beta = 2/3$.

5 Phenomenology

In this section we will give a detailed interpretation of the numerical results we obtained in the previous section. First we discuss an $SO(10)$ GUT models with a minimal spectrum. After turning on the flux F_X , the $SO(10)$ gauge group is broken to $SU(5) \times U(1)_X$, which will be interpreted as a flipped $SU(5)$ GUT from which further gauge breaking to the MSSM is possible.

5.1 $SO(10)$ GUT

The examples we presented in the previous section have the following general spectrum:

Matter	Rep.	generation	$U(1)_C$
$\mathbf{16}_M$	$\mathbf{16}^{(3)}$	3	1
$\mathbf{10}_H$	$\mathbf{10}^{(1,3)}$	1	-2
$\mathbf{10}_H$	$\mathbf{10}^{(1,3)}$	$k-1$	-2
-	$\mathbf{10}^{(3,3)}$	$-k$	2

(127)

Here k is the number of generations on the $\mathbf{10}$ curve, as given in the above examples. The $U(1)_C$ is of the Cartan subalgebra of $SU(4)_\perp$ that is not removed by the monodromy, as discussed in [8]. The Yukawa coupling is filtered by the conservation of this $U(1)_C$. The superpotential is:

$$W \supset \mathbf{16}_1 \mathbf{16}_1 \mathbf{10}_{-2} + \dots \quad (128)$$

This model only satisfies the minimum requirement of a three generation $SO(10)$ GUT model. Some Higgs fields, such as $\mathbf{210}$, $\mathbf{120}$, and $\mathbf{126} + \overline{\mathbf{126}}$ that break the $SO(10)$ gauge group to the $SU(5)$ and MSSM gauge groups, are absent in the F-theory construction. There are two possible ways to break the gauge group. One is to break to MSSM-like models by introducing a non-abelian instanton (flux) which can be further broken into a product of $U(1)$'s [3, 50]. The other possibility is to introduce abelian flux of the form [3, 48] $[F_X] = E_i - E_j$ to break $SO(10)$ to $SU(5) \times U(1)_X$ ¹⁰. This is discussed in detail below.

5.2 A flipped $SU(5)$ interpretation

5.2.1 Restriction of F_X

When the abelian flux F_X is turned on, the breaking pattern of the $SO(10)$ gauge group is:

$$\begin{aligned}
SO(10) &\supset SU(5) \times U(1)_X \\
\mathbf{45} &\rightarrow \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_4 + \overline{\mathbf{10}}_{-4} \\
\mathbf{16} &\rightarrow \mathbf{10}_{-1} + \overline{\mathbf{5}}_3 + \mathbf{1}_{-5} \\
\mathbf{10} &\rightarrow \mathbf{5}_2 + \overline{\mathbf{5}}_{-2}
\end{aligned} \quad (129)$$

Since we do not need $\mathbf{10}_4$ and $\overline{\mathbf{10}}_{-4}$ from the adjoint representation in the spectrum, we require their chirality to be equal to zero. Hence if L_X is the line bundle associated with F_X ,

¹⁰In the global $U(1)$ -restricted Tate model the GUT gauge group is $SO(10) \times U(1)_C$, therefore it breaks into $SU(5) \times U(1)_X \times U(1)_C$ after the flux is turned on. Since $U(1)_C$ is used to confine the Yukawa couplings of the $SO(10)$ curves, in what follows we focus the discussion on the phenomenology of the $SU(5) \times U(1)_X$ gauge group.

we set $\chi(S, L_X^4) = 0$ and $\chi(S, L_X^{-4}) = 0$. Then [3]:

$$L_X = \mathcal{O}_S(E_i - E_j)^{1/4}, \quad (130)$$

where E_i are the exceptional divisors of dP_k . L_X will change the chirality of the matter fields on each curve if F_X restricts non-trivially to that curve, i.e. if $F_X|_\Sigma = c_1(F_X) \cdot_S \Sigma \neq 0$. The net matter chirality on the curves in terms of the bundles can be summarized as:

$$n_{\mathbf{10}_{-1}} = -\chi(\Sigma_{\mathbf{16}}, K_{\Sigma_{\mathbf{16}}}^{1/2} \otimes V \otimes L_X^{-1}|_{\Sigma_{\mathbf{16}}}), \quad (131)$$

$$n_{\bar{\mathbf{5}}_3} = -\chi(\Sigma_{\mathbf{16}}, K_{\Sigma_{\mathbf{16}}}^{1/2} \otimes V \otimes L_X^3|_{\Sigma_{\mathbf{16}}}), \quad (132)$$

$$n_{\mathbf{1}_{-5}} = -\chi(\Sigma_{\mathbf{16}}, K_{\Sigma_{\mathbf{16}}}^{1/2} \otimes V \otimes L_X^{-5}|_{\Sigma_{\mathbf{16}}}), \quad (133)$$

$$n_{\mathbf{5}_2} = -\chi(\Sigma_{\mathbf{10}}, K_{\Sigma_{\mathbf{10}}}^{1/2} \otimes \wedge^2 V \otimes L_X^2|_{\Sigma_{\mathbf{10}}}). \quad (134)$$

We can calculate the net chiralities n_Σ by using $\chi(\Sigma, K_\Sigma^{1/2} \otimes V \otimes L|_\Sigma) = \deg(V \otimes L|_\Sigma)$, and the formula

$$\deg(V \otimes L) = c_1(V \otimes L) = c_1(V) + c_1(L^r), \quad (135)$$

where r is the rank of V . Since L_X is fractional and only L_X^4 makes sense physically, we require that $L_X^{1/4} \otimes V_{16}$ takes the value n_Σ on Σ , i.e. $(L_X^{1/4} \otimes V)|_\Sigma = n_\Sigma$, while we set $L_X^4|_\Sigma = N_\Sigma$ [18, 51, 11]. We summarize the modified chiralities on the matter curves with non-trivial restrictions of F_X in Table 9.

Curve	Matter	Chirality	Model 5B	rest. F_X
$\mathbf{16}_{-3}^{(1)}$	$\mathbf{10}_{-3,-1}$	$n_{\mathbf{16}}^{(1)}$	0	1
	$\bar{\mathbf{5}}_{-3,3}$	$n_{\mathbf{16}}^{(1)} + N_{\mathbf{16}}^{(1)}$	0 + 1	
	$\mathbf{1}_{-3,-5}$	$n_{\mathbf{16}}^{(1)} - N_{\mathbf{16}}^{(1)}$	0 - 1	
$\mathbf{16}_1^{(3)}$	$\mathbf{10}_{1,-1}$	$n_{\mathbf{16}}^{(3)}$	4	-1
	$\bar{\mathbf{5}}_{1,3}$	$n_{\mathbf{16}}^{(3)} + N_{\mathbf{16}}^{(3)}$	4 - 1	
	$\mathbf{1}_{1,-5}$	$n_{\mathbf{16}}^{(3)} - N_{\mathbf{16}}^{(3)}$	4 + 1	
$\mathbf{10}_{-2}^{(1,3)}$	$\bar{\mathbf{5}}_{-2,2}$	$n_{\mathbf{10}}^{(1,3)} + N_{\mathbf{10}}^{(1,3)}$	8 + 0	0
	$\bar{\mathbf{5}}_{-2,-2}$	$n_{\mathbf{10}}^{(1,3)}$	8	
$\mathbf{10}_2^{(3,3)}$	$\bar{\mathbf{5}}_{2,2}$	$n_{\mathbf{10}}^{(3,3)} + N_{\mathbf{10}}^{(3,3)}$	-8 + 0	0
	$\bar{\mathbf{5}}_{2,-2}$	$n_{\mathbf{10}}^{(3,3)}$	-8	

Table 9: The spectrum after $[F_X] = E_2 - E_3$ is turned on and restricted to the matter curves, with the numerical results from the example of Model 5B. The first number of the subscription of the matter representation is the $U(1)_C$ charge and the second is the $U(1)_X$ charge.

5.2.2 An example of a flipped $SU(5)$ GUT

In Example 5A we choose for F_X the class $F_X = E_1 - E_2$ such that there are trivial restrictions to all the curves. Thus the chirality on each curve remains unchanged. After the flux is turned

on, we can interpret the spectrum as that of the flipped $SU(5)$ model:

Matter	rep.	generation
$\mathbf{10}_M$	$\mathbf{10}_{1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{1,-5}$	3
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,-2}$	1
$\mathbf{10}_H + \bar{\mathbf{10}}_H$	$\mathbf{10}_{1,-1} + \bar{\mathbf{10}}_{1,1}$	1
$\mathbf{5} + \bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{-2,2} + \bar{\mathbf{5}}_{-2,-2}$ $\mathbf{5}_{2,2} + \mathbf{5}_{2,-2}$	1 -2

(136)

Although the advantage of this choice is that there is no exotic fermion and the quantum numbers of the matter are typical, the superheavy Higgses $\mathbf{10}_H$ and $\bar{\mathbf{10}}_H$ which are needed for breaking the gauge group to the MSSM are not obvious from the spectrum. We may claim that they are a vector-like pair from the $\mathbf{16}^{(3)}$ curve, but we are not able to fix the number such of pairs. Therefore, we propose a configuration where the flux F_X restricts non-trivially to both the 16 curves.

Consider $[F_X] = E_2 - E_3$ in Example 5B. The flux then takes the value 1 on $\mathbf{16}^{(1)}$ and -1 on $\mathbf{16}^{(3)}$. Therefore it will reduce the generation number of $\bar{\mathbf{5}}$ representation from curve $\mathbf{16}^{(3)}$ by one. That is the reason we consider Example 5B as a four-generation model. The detailed effect of this flux can be also found in Table 9. We conclude that the flipped $SU(5)$ spectrum of Example 5B is:

Matter	rep.	generation
$\mathbf{10}_M$	$\mathbf{10}_{1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{1,-5}$	3
$\mathbf{10}_H + \bar{\mathbf{10}}_H$	$\mathbf{10}_{1,-1} + \bar{\mathbf{10}}_{1,1}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,-2}$	1
$\mathbf{10}$	$\mathbf{10}_{1,-1}$	1
$\bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{-3,3}$	1
$\mathbf{1}$	$\mathbf{1}_{3,5}$	1
$\mathbf{1}$	$\mathbf{1}_{1,-5}$	2
$\mathbf{5} + \bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{-2,2} + \bar{\mathbf{5}}_{-2,-2}$ $\mathbf{5}_{2,2} + \mathbf{5}_{2,-2}$	7 -8

(137)

The Yukawa couplings are standard:

$$\begin{aligned}
W \supset & \mathbf{10}_{1,-1M} \mathbf{10}_{1,-1M} \bar{\mathbf{5}}_{-2,2h} + \mathbf{10}_{1,-1M} \bar{\mathbf{5}}_{1,3M} \bar{\mathbf{5}}_{-2,-2h} + \bar{\mathbf{5}}_{1,3M} \mathbf{1}_{1,-5M} \bar{\mathbf{5}}_{-2,2h} \\
& + \mathbf{10}_{1,-1H} \mathbf{10}_{1,-1H} \bar{\mathbf{5}}_{-2,2h} + \bar{\mathbf{10}}_{1,1H} \bar{\mathbf{10}}_{1,1H} \bar{\mathbf{5}}_{-2,-2h} + \dots
\end{aligned}
\tag{138}$$

Again the $\mathbf{10}_H$ and $\bar{\mathbf{10}}_H$ Higgs fields have to come from $\mathbf{16}^{(3)}$, which is not obvious. Although the flux can reduce the chirality of one field, it can increase the chirality of another field. The total effect shows that some exotic fields from the $\mathbf{16}$ curves are unavoidable at the current stage.

5.2.3 The Singlet Higgs

In the $SU(5)$ spectral cover the singlet matter is not obvious. A semi-local approach to this problem suggests that the singlet fields localize on $\lambda_i = \lambda_j$ for $i \neq j$ [11]. In the Georgi-Glashow $SU(5)$ the singlet is taken as the right-handed neutrino, while in the flipped $SU(5)$ model this singlet is interpreted as the right-handed electron which must clearly be included. Since in our discussion the model starts from an $SO(10)$ gauge group and is then broken to $SU(5)$, the matter singlet is naturally embedded into the **16** representation. Therefore we may avoid the singlet in the $SU(4)$ spectral cover setup.

On the other hand, in order to explain the neutrino mass problem by a seesaw mechanism, there is a Yukawa coupling term in the superpotential that completes the flipped $SU(5)$ model. This Yukawa coupling term is [52, 53]:

$$\mathbf{10}_{-1M} \overline{\mathbf{10}}_{1H} \mathbf{1}_0 \phi. \quad (139)$$

This singlet $\mathbf{1}_0$ can be found neither on the **16** nor on the **10** curves we have discussed. It is an $SO(10)$ object, which can be identified in the $SU(4)$ cover with the locus given by $\prod_{i < j} (\lambda_i - \lambda_j) = 0$. Since it is antisymmetric, we can square it to make it symmetric [11]. To calculate its matter chirality we need to compute the genus of the curve and the degree of the line bundle. However the mechanism is still not clear, and we hope to obtain a more global understanding of this singlet curve along the lines of [11] in the future. Therefore, here we just assume that this singlet exists and provides the above Yukawa coupling.

6 Conclusions and outlook

In this paper we have explicitly constructed a class of global $SO(10)$ F-theory models. We have shown that toric geometry makes it possible to obtain a large number of Calabi-Yau fourfolds which are elliptic fibrations over non-Fano base manifolds. Inside the base manifolds we could identify a large number of divisors where one can construct GUT models on. These divisors are del Pezzo and satisfy a (mathematical and/or physical) decoupling limit. We also found that many of the base manifolds we have constructed satisfy the definitions of almost Fano manifolds. Constructing the elliptically fibered fourfolds further reduces the number of possible global models since not all base geometries can be extended to geometries of fourfolds that are torically realized as reflexive polyhedra with the right nef partition. Despite these issues we have managed to construct a significant number of geometries which are suitable for non-trivial global F-theory GUT models.

The second goal of our paper was to construct $SO(10)$ GUT models. We worked out several examples in different geometric setups in detail. We factorized the spectral cover in order to obtain non-zero generation number on the **10** matter curve. This gives us a Higgs field in the $SO(10)$ GUT, as well as further degrees of freedom which can be tuned to get more realistic models. By turning on the massless gauge flux F_X we break the $SO(10)$ gauge group to $SU(5) \times U(1)_X$. This can be interpreted as a flipped $SU(5)$ GUT model. Superheavy Higgs fields can be identified in the spectrum which implies that this GUT model can be broken to the MSSM by the associated Higgs mechanism.

There are several directions for further research. Firstly, it would be interesting to construct more general fourfolds. Our approach was to first construct a base manifold and then the elliptic fibrations. All the base manifolds were obtained from point and curve blowups in

Fano threefolds which are hypersurfaces in \mathbb{P}^4 . It can be expected that the models one gets from this rather restricted class may have very similar properties. It would be interesting to investigate the F-theory models one gets from more general base manifolds. Such a task will require an extensive computer search for models. This may be useful for the discussion of an “F-theory landscape”.

A second issue which we have not touched at all in this paper is the problem of moduli stabilization. We have shown in explicit examples that it is possible to find three generation models in our geometries. However, saying that one can tune the moduli to specific values in order to get interesting physical properties does not imply that this necessarily happens. Finding a way to stabilize the moduli in explicit examples is a crucial requirement for the success of global F-theory GUTs. Related to this issue is the calculation of superpotentials and instanton corrections in F-theory. This has been recently discussed in [54, 55, 56, 57, 58, 59] from various viewpoints. In a different context the problem of computing the superpotential has been recently addressed in [60, 61, 62, 63] where the calculation of F-theory superpotentials has been related to open string mirror symmetry. It would be interesting to investigate whether this approach can also be applied to the F-theory models we have considered here.

In our discussion of the $SO(10)$ models we relied on a split spectral cover in order to produce chiral matter on the **10** curves. Recently, it has been pointed out in [64] that the split spectral cover may not be well-defined globally. In $SU(5)$ models this leads to a possible generation of degree 4 proton decay operators despite the introduction of a split spectral cover. A similar problem may leave us without chiral matter on the **10** curves in the $SO(10)$ models. Therefore an analysis along the lines of [64] should be done also for $SO(10)$ F-theory GUTs.

In order to discuss tadpole cancellation we have calculated the Euler number for the Calabi-Yau fourfolds after a crepant resolution of the $SO(10)$ singularities. We have compared our results with a conjectured formula for the Euler number given in [10]. For three out of the five examples we have discussed we found a mismatch between the two ways of calculation. So far we have not been able to pin down where the mismatch in the results is rooted. An obvious explanation is that some of the assumptions under which the formula in [10] was claimed to hold were violated. One possibility would be that there are more non-abelian enhancements in the Weierstrass model than just the $SO(10)$. Evidence for this can be collected by calculating $h^{1,1}$ of the fourfold. In the examples with the discrepancy in the Euler numbers it can be shown that after the resolution of the $SO(10)$ singularities $h^{1,1}$ changes by more than the rank of $SO(10)$ ¹¹. However, there can also be further reasons which can contribute to a discrepancy in the Euler numbers. One further possibility may be that the fourfold geometries exhibit terminal singularities. Given the recent results of [64] one may also speculate whether the discrepancy in the Euler numbers is due to a globally ill-defined spectral cover, at least for models without a heterotic dual. One possibility to collect evidence for this is to check whether the models for which the Euler numbers match actually have a heterotic dual. For this to hold the fourfold Y has to admit a $K3$ fibration over S_{GUT} [16]. For the model with the GUT brane on dP_7 in [10] and the dP_5 model discussed in [12] as well as our model 5 we indeed found such a fibration structure, which indicates that these models do have heterotic duals. We intend to explain and resolve this Euler number discrepancy in the future.

While we were preparing this paper for publication [65] appeared where it was discussed that the decoupling of gravity in the four-dimensional theory implies that the GUT brane

¹¹We thank T. Grimm for pointing this out to us.

	y_1	y_2	y_3	y_4	y_5	Σ
w_1	1	1	1	1	1	5

Table 10: Weight vector for $\mathbb{P}^4[d]$.

wraps a non-commutative four cycle. It would be interesting to refine our discussion of decoupling limit by taking into account non-commutativity.

A List of geometries

In this appendix we provide a class of base manifolds which come from up to three blowups of curves and points in $\mathbb{P}^4[d]$ with $d = 2, 3, 4$, and have at least one del Pezzo divisor with a physical or mathematical decoupling limit. In the specification of the base manifold we restrict ourselves to those models where the degrees of the hypersurface equation which determine the base are strictly smaller than the sums of the weights. This class includes all examples with up to three blowups which have been discussed previously in the literature. For technical reasons we will only allow for weight matrices which do not lead to further weight vectors in the lattice polytope they create. For the study of the decoupling limit it is convenient to look only at models where the number of generators of the Mori cone is the same as the number of Kähler moduli. This technical requirement does not significantly reduce the number of models.

We have examined 241 base geometries in total. 208 of these geometries have at least one divisor which is del Pezzo and is subject to a mathematical and/or physical decoupling limit. For all the models we have checked the 'almost Fano' property and whether it is possible to construct an elliptically fibered Calabi-Yau fourfold which is characterized by a reflexive polyhedron in toric geometry. For 86 of the models we find a reflexive polyhedron for the fourfold. We will explicitly give the data of the geometries which satisfy the requirements of an 'almost Fano' base and/or reflexivity of the fourfold polyhedron.

A.1 Weight matrices

We will now discuss the weight matrices for the models we would like to construct. These weight matrices together with the specification of the hypersurface divisor of the base B encode all the data we need for our calculations. Since the ambient space of the three examples of Fano threefolds is always \mathbb{P}^4 the weight matrices will be the same for each Fano. Only the degrees of the hypersurface equations specifying the base will change. Therefore we can discuss the weight matrices for all three Fanos at once.

The weight vector of \mathbb{P}^4 is given in table 10. Let us first consider curve blowups. Since all variables have the same weight the choice is unique up to permutation of variables. The weight vector is given in table 11. For $\mathbb{P}^4[3]$ and $\mathbb{P}^4[4]$ it is possible to tune the complex structure moduli in such a way that there is a singularity at $(0, 0, 0, y_4, y_5)$ [10].

For our models to have a decoupling limit it is usually not enough to blow up just one curve. Taking into account the symmetries, there are four possibilities to blow up a second curve, as shown in table 12. Let us discuss these four weight vectors in turn. It looks like $w_{3,1}$ comes from a singular transition at $(y_1, y_2, 0, 0, 0)$. However, looking at table 2 we find that $y_4 = y_5 = y_6 = 0$ is in the Stanley-Reisner ideal, so this point actually has to be excluded.

	y_1	y_2	y_3	y_4	y_5	y_6	Σ
w_1	1	1	1	1	1	0	5
w_2	0	0	0	1	1	1	3

Table 11: Blowup of the first curve.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	Σ
$w_{3,1}$	1	1	0	0	0	0	1	3
$w_{3,2}$	0	1	0	1	0	0	1	3
$w_{3,3}$	0	1	0	0	0	1	1	3
$w_{3,4}$	0	0	0	1	0	1	1	3

Table 12: Possibilities to blow up a second curve.

The weight vector $w_{3,2}$ describes the second curve blowup in [12]. It comes from the blowup of the singularity at $(0, y_2, 0, y_4, 0, 0)$. Note however that we do not insist on singular transitions in our models. In our further discussion we will not consider the weight vectors $w_{3,3}$ and $w_{3,4}$ because the polytope generated by these weight vectors has more (for $w_{3,3}$) or fewer (for $w_{3,4}$, where the same curve is blown up twice) weight vectors than those given by the weight matrix.

We also consider blowups of three curves. The weight matrix for this setup consists of $w_1, w_2, w_{3,1}, w_{3,2}$, as shown in table 13.

Apart from blowing up curves, we can also make point blowups. If we blow up one curve and one point there are two inequivalent possibilities summarized in table 14. Every ‘*’- or ‘ \diamond ’-entry in the tables stands for one possible position of a 1. Since we want a point blowup only one of the ‘*’- or ‘ \diamond ’-entries can be set to one while the others must be set to zero. The difference between the ‘*’- and ‘ \diamond ’-entries will be explained below. Setups which lead to additional weight vectors are again excluded. Blowing up one curve and two points, we can distinguish between two cases, coming from the two possibilities of blowing up one curve and one point. Altogether we find four possible structures given in tables 15 and 16. If we blow up two curves and one point we can distinguish between the cases where we choose $w_{3,1}$ or $w_{3,2}$ from table 12 as our second curve. The inequivalent possibilities we have are listed in tables 17 and 18. Finally, we can also consider cases where we do not blow up any curves but up to three points. Given the symmetry of our setup and the restrictions we imposed there is only one possibility to blow up a single point. The corresponding weight matrix is given in table 19. If we blow up two points there is only one weight matrix which meets our

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	Σ
w_1	1	1	1	1	1	0	0	0	5
w_2	0	0	0	1	1	1	0	0	3
w_3	1	1	0	0	0	0	1	0	3
w_4	0	1	0	1	0	0	0	1	3

Table 13: Blowup of three curves.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	\sum
w_1	1	1	1	1	1	0	0	5
w_2	0	0	0	1	1	1	0	3
w_3	*	0	0	*	0	0	1	2

Table 14: Blowups of one curve and one point.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	\sum
w_1	1	1	1	1	1	0	0	0	5
w_2	0	0	0	1	1	1	0	0	3
w_3	1	0	0	0	0	0	1	0	2
w_4	0	*	0	*	0	0	0	1	2

Table 15: Blowups of one curve and two points, first case.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	\sum
w_1	1	1	1	1	1	0	0	0	5
w_2	0	0	0	1	1	1	0	0	3
w_3	0	0	0	1	0	0	1	0	2
w_4	\diamond	0	0	0	*	0	0	1	2

Table 16: Blowups of one curve and two points, second case.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	\sum
w_1	1	1	1	1	1	0	0	0	5
w_2	0	0	0	1	1	1	0	0	3
w_3	1	1	0	0	0	0	1	0	3
w_4	*	0	0	\diamond	0	0	0	1	2

Table 17: Blowups of two curves and one point, first case.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	\sum
w_1	1	1	1	1	1	0	0	0	5
w_2	0	0	0	1	1	1	0	0	3
w_3	0	1	0	1	0	0	1	0	3
w_4	*	*	0	*	\diamond	0	0	1	2

Table 18: Blowups of two curves and one point, second case.

	y_1	y_2	y_3	y_4	y_5	y_6	\sum
w_1	1	1	1	1	1	0	5
w_2	1	0	0	0	0	1	2

Table 19: Blowup of one point.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	\sum
w_1	1	1	1	1	1	0	0	5
w_2	1	0	0	0	0	1	0	2
w_3	0	1	0	0	0	0	1	2

Table 20: Blowup of two points.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	\sum
w_1	1	1	1	1	1	0	0	0	5
w_2	1	0	0	0	0	1	0	0	2
w_3	0	1	0	0	0	0	1	0	2
w_4	0	0	1	0	0	0	0	1	2

Table 21: Blowup of three points.

requirements. This is shown in table 20. If we blow up three points it turns out that there is also only one possibility, which is given in table 21.

So far, we have built up our weight matrices line by line and have taken into account symmetries which come from the exchange of columns in the weight matrices because this only amounts to a permutation of coordinates. However, there are further symmetries which come from a combined exchange of rows and columns in the weight matrices. This leads to redundancies in the weight matrices given in the tables. We have marked the redundant choices by a ' \diamond '. Altogether there are three pairs of weight matrices that are equivalent: The second choice of weights in table 15 can be transformed into the first choice of weights in table 16 by exchanging the last two rows and permuting the columns. Similar manipulations transform the first choice of weights in table 17 into the second choice in this table. Furthermore in table 18 the choice with the 1 in the last line positioned in the second column can be transformed into the last configuration where the 1 is in the fifth position, by exchanging the second and the third row and permuting the columns. Due to these extra symmetries of the weight matrices, one can always place a '0' at every position where a ' \diamond ' entry appears.

There may also arise additional symmetries from the choice of degrees in the hypersurface equations. Furthermore different triangulations of the ambient space may also lead to the same results. We have not removed these redundancies in the tables below.

A.2 Base manifolds and GUT divisors

We will now discuss specific base geometries. Since the number of base manifolds we have constructed is quite large we will only list those geometries which satisfy the 'almost Fano' property and/or where the associated fourfold is characterized by a reflexive polytope. We will organize the data into tables with the following entries.

1. **Weights:** This specifies one of the weight matrices listed above, formatted as $nCmPk$, where n and m are the number of blown up curves and points, respectively. k indicates the position in the list of weight matrices.
2. **Triangulation:** For some weight matrices the N-lattice polytope does not define the ambient space uniquely. The one-cones given by all the points of the N-lattice polytope

may realized by different fans. This entry in the tables labels different triangulations of the polytopes, that is, the different fans.

3. **Base:** We give a list of degrees which specify the base manifold B . The ordering is given by the ordering of the weight vectors. Whenever the 'almost Fano' criterion is satisfied we will add a $()^*$ to the vector of degrees. If the polytope of the associated Calabi-Yau fourfold is reflexive we will add a $()^\circ$.
4. **GUT divisor:** This lists the divisors in a given model which are del Pezzo surfaces satisfying a physical and/or mathematical decoupling limit. If a particular model has been discussed in detail in Section 4 we mark this by (Mx) (x stands for the number of the model) next to the corresponding GUT divisor.
5. **Genus:** Here we give the genera of the matter curves for $SO(10)$ models¹², ordered as (g_{10}, g_{16}) .
6. **Yukawa:** This gives the number of Yukawa couplings for $SO(10)$ models ordered as $(n_{E_7}, n_{SO(14)})$.
7. **Decoupling:** This entry indicates whether there is a mathematical ('m') or physical ('p') decoupling limit.

A.2.1 $\mathbb{P}^4[4]$

Blowup of 1 curve

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C0P1	1	$(4, 2)^{\circ*}$	dP_7	$(2, 6)$	$(10, 44)$	m

Blowup of 2 curves

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
2C0P1	1	$(4, 2, 2)^{\circ*}$	dP_7	$(2, 6)$	$(10, 44)$	m
			dP_7	$(2, 6)$	$(10, 44)$	m
2C0P2	1	$(4, 2, 2)^{\circ*}$	dP_5 (M1)	$(2, 8)$	$(14, 68)$	mp
2C0P2	2	$(4, 2, 2)^{\circ*}$	dP_5	$(2, 8)$	$(14, 68)$	mp

Blowup of 3 curves

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
3C0P1	1	$(4, 2, 2, 2)^{\circ*}$	dP_3	$(2, 10)$	$(18, 92)$	mp
			dP_1	$(6, 20)$	$(38, 164)$	m
3C0P1	2	$(4, 2, 2, 2)^*$	dP_1	$(6, 20)$	$(38, 164)$	m
			dP_5	$(2, 8)$	$(14, 68)$	m
3C0P1	3	$(4, 2, 2, 2)^*$	dP_5	$(2, 8)$	$(14, 68)$	m
			dP_1	$(6, 20)$	$(38, 164)$	m

¹²We give $SO(10)$ specific data here, but one can of course also construct $SU(5)$ models on these geometries.

Blowup of 1 curve and 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C1P1	2	$(4, 2, 1)^*$	dP_7	$(2, 6)$	$(10, 44)$	m
			dP_6	$(1, 4)$	$(16, 36)$	mp
1C1P2	1	$(4, 2, 1)^{*\circ}$	dP_4 (M3)	$(2, 9)$	$(16, 80)$	mp

Blowup of 1 curve and 2 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C2P3	1	$(4, 2, 1, 1)^{*\circ}$	dP_1	$(2, 12)$	$(22, 116)$	mp

Blowup of 2 curves and 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
2C1P1	1	$(4, 2, 2, 1)^*$	dP_4	$(2, 9)$	$(16, 80)$	mp
			dP_7	$(2, 6)$	$(10, 44)$	m
2C1P3	1	$(4, 2, 2, 1)^*$	dP_2	$(2, 11)$	$(20, 104)$	mp
2C1P3	2	$(4, 2, 2, 1)^*$	dP_5	$(2, 8)$	$(14, 68)$	mp
2C1P4	1	$(4, 2, 2, 1)^{*\circ}$	dP_5	$(2, 8)$	$(14, 68)$	m
			dP_5	$(1, 5)$	$(8, 48)$	mp
			dP_5 (M4)	$(2, 8)$	$(14, 68)$	mp

Blowup of 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C1P1	1	$(4, 1)^{*\circ}$	dP_6 (M2)	$(1, 4)$	$(6, 36)$	mp

A.2.2 $\mathbb{P}^4[3]$

Blowup of 1 curve

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C0P1	1	$(3, 2)^{*\circ}$	dP_1	$(6, 20)$	$(38, 164)$	m
1C0P1	1	$(3, 1)^{*\circ}$	dP_4	$(3, 12)$	$(22, 100)$	m

2 curves

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
2C0P1	1	$(3, 1, 2)^{*\circ}$	dP_4	$(3, 12)$	$(22, 100)$	m
			dP_1	$(9, 26)$	$(50, 200)$	m
			dP_1	$(6, 20)$	$(38, 164)$	m
2C0P1	1	$(3, 2, 2)^{*\circ}$	dP_1	$(6, 20)$	$(38, 164)$	m
			dP_1	$(6, 20)$	$(38, 164)$	m

2C0P1	1	$(3, 2, 1)^{*o}$	dP_1	(6, 20)	(38, 164)	m
			dP_1	(9, 26)	(50, 200)	m
			dP_4	(3, 12)	(22, 100)	m
2C0P1	1	$(3, 1, 1)^{*o}$	dP_4	(3, 12)	(22, 100)	m
			dP_4	(3, 12)	(22, 100)	m
2C0P2	1	$(3, 1, 2)^*$	dP_5	(2, 8)	(14, 68)	mp
2C0P2	1	$(3, 2, 2)^{*o}$	dP_2	(5, 17)	(32, 140)	m
			dP_1	(5, 18)	(34, 152)	mp
2C0P2	1	$(3, 1, 1)^*$	dP_2	(3, 14)	(26, 124)	mp
2C0P2	2	$(3, 2, 2)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_2	(5, 17)	(32, 140)	m
2C0P2	2	$(3, 2, 1)^*$	dP_5	(2, 8)	(14, 68)	mp
2C0P2	2	$(3, 1, 1)^*$	dP_2	(3, 14)	(26, 124)	mp

Blowup of 3 curves

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
3C0P1	1	$(3, 2, 2, 2)^{*o}$	dP_2	(5, 17)	(32, 140)	m
			dP_1	(4, 16)	(30, 140)	mp
			dP_2	(5, 17)	(32, 140)	m
3C0P1	1	$(3, 2, 2, 1)^*$	dP_6	(1, 4)	(6, 36)	mp
			dP_3	(4, 14)	(26, 116)	m
3C0P1	1	$(3, 1, 2, 1)^*$	dP_3	(2, 10)	(18, 92)	mp
3C0P1	1	$(3, 2, 1, 1)^*$	dP_3	(2, 10)	(18, 92)	mp
3C0P1	1	$(3, 1, 1, 1)^{*o}$	dP_0	(3, 16)	(30, 48)	mp
3C0P1	2	$(3, 2, 2, 2)^{*o}$	dP_2	(5, 17)	(32, 140)	m
			dP_2	(4, 15)	(28, 128)	mp
3C0P1	2	$(3, 1, 2, 2)^*$	dP_2	(5, 17)	(32, 140)	m
			dP_1	(6, 19)	(36, 152)	m
			dP_5	(2, 8)	(14, 68)	m
3C0P1	2	$(3, 1, 2, 1)^*$	dP_2	(3, 14)	(26, 124)	m
3C0P1	3	$(3, 2, 2, 2)^{*o}$	dP_1	(5, 18)	(34, 152)	m
			dP_2	(4, 15)	(28, 128)	mp
			dP_2	(5, 17)	(32, 140)	m
3C0P1	3	$(3, 2, 1, 2)^*$	dP_5	(2, 8)	(14, 68)	m
			dP_2	(6, 19)	(36, 152)	m
			dP_2	(5, 17)	(32, 140)	m
3C0P1	3	$(3, 2, 1, 1)^*$	dP_2	(3, 14)	(26, 124)	m

Blowup of 1 curve and 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C1P1	1	$(3, 2, 1)^{*o}$	dP_3	(6, 18)	(34, 140)	m
			dP_4	(6, 16)	(30, 120)	p

			dP_3	(4, 14)	(26, 116)	p
1C1P1	2	$(3, 2, 1)^{*o}$	dP_1	(6, 20)	(38, 164)	m
			dP_1	(4, 16)	(30, 140)	mp
1C1P1	2	$(3, 1, 1)^*$	dP_4	(3, 12)	(22, 100)	m
			dP_1	(4, 16)	(30, 140)	mp
1C1P2	1	$(3, 2, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_2	(3, 13)	(24, 116)	mp
1C1P2	1	$(3, 1, 1)^{*o}$	dP_4	(2, 9)	(16, 80)	mp

Blowup of 1 curve and 2 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C2P1	1	$(3, 2, 1, 1)^{*o}$	dP_2	(4, 15)	(28, 128)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_1	(6, 20)	(39, 164)	m
1C2P1	1	$(3, 1, 1, 1)^*$	dP_2	(4, 15)	(28, 128)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_4	(3, 12)	(22, 100)	m
1C2P1	2	$(3, 2, 1, 1)^{*o}$	dP_2	(4, 15)	(28, 128)	mp
			dP_3	(5, 16)	(30, 128)	mp
			dP_4	(6, 16)	(30, 120)	p
			dP_3	(6, 18)	(34, 140)	m
1C2P1	3	$(3, 2, 1, 1)^{*o}$	dP_6	(7, 15)	(28, 100)	m
			dP_2	(4, 15)	(28, 128)	mp
			dP_3	(5, 16)	(30, 128)	mp
			dP_4	(6, 16)	(30, 120)	p
			dP_3	(6, 18)	(34, 140)	m
1C2P1	4	$(3, 2, 1, 1)^{*o}$	dP_4	(4, 13)	(24, 104)	p
			dP_4	(4, 13)	(24, 104)	p
			dP_1	(5, 18)	(34, 152)	mp
			dP_4	(6, 16)	(30, 120)	p
			dP_4	(6, 16)	(30, 120)	p
1C2P2	1	$(3, 2, 1, 1)^{*o}$	dP_4	(6, 16)	(30, 120)	p
			dP_2	(5, 17)	(32, 140)	p
			dP_4	(6, 16)	(30, 120)	p
			dP_3	(4, 14)	(26, 116)	p
			dP_3	(3, 12)	(22, 104)	p
1C2P2	2	$(3, 2, 1, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_3	(3, 12)	(22, 104)	mp
1C2P2	2	$(3, 1, 1, 1)^*$	dP_4	(2, 9)	(16, 80)	p
			dP_2	(4, 15)	(28, 128)	mp
1C2P3	1	$(3, 2, 1, 1)^{*o}$	dP_2	(3, 13)	(24, 116)	mp
			dP_1	(4, 16)	(30, 140)	mp
			dP_2	(3, 13)	(24, 116)	mp
1C2P3	1	$(3, 1, 1, 1)^{*o}$	dP_4	(1, 6)	(10, 60)	mp

Blowup of 2 curves and 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
2C1P1	1	$(3, 2, 2, 1)^{\ast\circ}$	dP_1	(5, 18)	(34, 152)	mp
			dP_2	(3, 13)	(24, 116)	mp
			dP_1	(6, 20)	(38, 164)	m
2C1P1	1	$(3, 1, 2, 1)^{\ast}$	dP_1	(5, 18)	(34, 152)	mp
			dP_2	(3, 13)	(24, 116)	mp
			dP_4	(3, 12)	(22, 100)	m
2C1P1	1	$(3, 2, 1, 1)^{\ast\circ}$	dP_4 (M5)	(2, 9)	(16, 80)	mp
			dP_1	(6, 20)	(38, 164)	m
2C1P1	1	$(3, 1, 1, 1)^{\ast\circ}$	dP_4	(2, 9)	(16, 80)	mp
			dP_4	(3, 12)	(22, 100)	m
2C1P1	2	$(3, 2, 2, 1)^{\ast\circ}$	dP_1	(5, 18)	(34, 152)	mp
			dP_4	(3, 11)	(20, 92)	p
			dP_4	(4, 13)	(24, 104)	p
			dP_4	(6, 16)	(30, 120)	p
			dP_3	(6, 18)	(34, 140)	m
2C1P1	2	$(3, 2, 1, 1)^{\ast\circ}$	dP_4	(2, 9)	(16, 80)	mp
			dP_1	(7, 22)	(42, 176)	mp
			dP_4	(6, 16)	(30, 120)	p
			dP_3	(6, 18)	(34, 140)	m
2C1P1	2	$(3, 1, 1, 1)^{\circ}$	dP_4	(2, 9)	(16, 80)	mp
			dP_1	(9, 25)	(48, 192)	p
			dP_4	(5, 15)	(28, 116)	m
2C1P2	1	$(3, 2, 2, 1)^{\ast\circ}$	dP_1	(6, 20)	(38, 164)	m
			dP_2	(3, 13)	(24, 116)	mp
			dP_1	(5, 18)	(34, 152)	mp
2C1P2	1	$(3, 1, 2, 1)^{\ast\circ}$	dP_1	(6, 20)	(38, 164)	m
			dP_4	(2, 9)	(16, 80)	mp
2C1P2	1	$(3, 2, 2, 1)^{\ast\circ}$	dP_4	(6, 16)	(30, 120)	p
			dP_4	(5, 15)	(28, 116)	m
			dP_5	(5, 16)	(30, 128)	mp
			dP_2	(4, 15)	(28, 128)	mp
2C1P2	2	$(3, 2, 2, 1)^{\ast\circ}$	dP_4	(6, 16)	(30, 120)	p
			dP_3	(5, 16)	(30, 128)	mp
			dP_4	(5, 15)	(28, 116)	m
			dP_2	(4, 15)	(28, 128)	mp
2C1P2	3	$(3, 2, 2, 1)^{\ast\circ}$	dP_4	(6, 16)	(30, 120)	p
			dP_1	(5, 18)	(34, 152)	mp
			dP_4	(5, 15)	(28, 116)	m
			dP_3	(4, 14)	(26, 116)	p
			dP_4	(4, 13)	(24, 104)	p
2C1P2	3	$(3, 2, 1, 1)^{\ast}$	dP_4	(6, 16)	(30, 120)	p
			dP_5	(2, 8)	(14, 68)	mp
			dP_3	(4, 14)	(26, 116)	p

			dP_0	(7, 23)	(44, 188)	p
2C1P2	4	$(3, 2, 2, 1)^{*o}$	dP_4	(6, 16)	(30, 120)	p
			dP_4	(5, 15)	(28, 116)	m
			dP_1	(5, 18)	(34, 152)	mp
			dP_3	(4, 14)	(26, 116)	p
			dP_4	(4, 13)	(24, 104)	p
2C1P2	4	$(3, 1, 2, 1)^*$	dP_4	(6, 16)	(30, 120)	p
			dP_5	(2, 8)	(14, 68)	mp
			dP_3	(4, 14)	(26, 116)	p
			dP_0	(7, 23)	(44, 188)	p
2C1P3	1	$(3, 2, 2, 1)^{*o}$	dP_2	(5, 17)	(32, 140)	m
			dP_1	(4, 16)	(30, 140)	mp
			dP_2	(3, 13)	(24, 116)	mp
2C1P3	1	$(3, 2, 1, 1)^*$	dP_5	(1, 5)	(8, 48)	mp
2C1P3	1	$(3, 1, 1, 1)^*$	dP_2	(2, 11)	(20, 104)	mp
2C1P3	2	$(3, 2, 2, 1)^{*o}$	dP_1	(5, 18)	(34, 151)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_2	(3, 13)	(24, 116)	mp
2C1P3	2	$(3, 1, 2, 1)^*$	dP_5	(2, 8)	(14, 68)	mp
			dP_2	(3, 13)	(24, 116)	mp
2C1P3	2	$(3, 1, 1, 1)^*$	dP_2	(3, 14)	(26, 124)	mp
2C1P3	3	$(3, 2, 2, 1)^{*o}$	dP_5	(5, 13)	(24, 96)	p
			dP_3	(5, 16)	(30, 128)	m
			dP_1	(4, 16)	(30, 140)	mp
			dP_5	(5, 13)	(24, 96)	p
			dP_3	(3, 12)	(22, 104)	p
2C1P3	3	$(3, 2, 1, 1)^*$	dP_5	(1, 5)	(8, 48)	mp
2C1P3	4	$(3, 2, 2, 1)^{*o}$	dP_5	(5, 13)	(24, 96)	p
			dP_5	(5, 17)	(32, 140)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_5	(5, 13)	(24, 96)	p
2C1P4	1	$(3, 2, 2, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_3	(2, 10)	(18, 92)	mp
			dP_1	(5, 18)	(34, 152)	mp
2C1P4	1	$(3, 2, 1, 1)^{*o}$	dP_5	(2, 8)	(14, 68)	mp
			dP_1	(3, 14)	(26, 128)	mp
			dP_2	(5, 17)	(32, 140)	mp
2C1P4	1	$(3, 1, 2, 1)^{*o}$	dP_2	(5, 17)	(32, 140)	mp
			dP_1	(3, 14)	(26, 128)	mp
			dP_5	(2, 8)	(14, 68)	mp

Blowup of 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C1P0	1	$(3, 1)^{*o}$	dP_1	(4, 16)	(30, 140)	mp

Blowup of 2 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C2P0	1	$(3, 1, 1)^{*o}$	dP_2	(4, 15)	(28, 128)	mp
			dP_2	(4, 15)	(28, 128)	mp

Blowup of 3 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C3P0	1	$(3, 1, 1, 1)^{*o}$	dP_3	(4, 14)	(26, 116)	p
			dP_3	(4, 14)	(26, 116)	p
			dP_3	(4, 14)	(26, 116)	p

A.2.3 $\mathbb{P}^4[2]$

Blowup of 1 curve

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C0P1	1	$(2, 1)^{*o}$	dP_1	(7, 22)	(42, 176)	m

Blowup of 2 curves

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
2C0P1	1	$(2, 1, 1)^{*o}$	dP_1	(7, 22)	(42, 176)	m
			dP_1	(8, 24)	(76, 188)	m
			dP_1	(7, 22)	(42, 176)	m
2C0P2	1	$(2, 1, 1)^{*o}$	dP_2	(6, 19)	(36, 152)	m
			dP_1	(6, 20)	(38, 164)	mp
2C0P2	2	$(2, 1, 1)^{*o}$	dP_1	(12, 30)	(58, 220)	p
			dP_1	(6, 20)	(38, 164)	mp
			dP_2	(6, 19)	(36, 152)	m

Blowup of 3 curves

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
3C0P1	1	$(2, 1, 1, 1)^{*o}$	dP_2	(6, 19)	(36, 152)	m
			dP_1	(5, 18)	(34, 152)	mp
			dP_1	(6, 20)	(38, 164)	m
3C0P1	2	$(2, 1, 1, 1)^{*o}$	dP_2	(6, 19)	(36, 152)	m
			dP_2	(5, 17)	(32, 140)	mp
			dP_1	(6, 20)	(38, 164)	m
			dP_1	(6, 20)	(38, 164)	m
3C0P1	3	$(2, 1, 1, 1)^{*o}$	dP_1	(6, 20)	(38, 164)	m
			dP_2	(5, 17)	(32, 140)	mp
			dP_1	(6, 20)	(38, 164)	m

			dP_2	(6, 19)	(36, 152)	m
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Blowup of 1 curve and 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C1P1	1	$(2, 1, 1)^{*o}$	dP_1	(9, 25)	(48, 192)	m
			dP_1	(6, 20)	(38, 164)	p
1C1P1	2	$(2, 1, 1)^{*o}$	dP_1	(7, 22)	(42, 176)	m
			dP_0	(6, 21)	(40, 176)	mp
1C1P2	1	$(2, 1, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_1	(5, 18)	(34, 152)	mp

Blowup of 1 curve and 2 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
1C2P1	1	$(2, 1, 1, 1)^{*o}$	dP_0	(6, 21)	(40, 176)	mp
			dP_0	(6, 21)	(40, 176)	mp
			dP_1	(7, 22)	(42, 176)	m
1C2P1	2	$(2, 1, 1, 1)^{*o}$	dP_0	(6, 21)	(40, 176)	mp
			dP_2	(6, 19)	(36, 152)	mp
			dP_1	(9, 25)	(48, 192)	mp
			dP_2	(7, 21)	(40, 164)	m
1C2P1	3	$(2, 1, 1, 1)^{*o}$	dP_0	(6, 21)	(40, 176)	mp
			dP_2	(6, 19)	(36, 152)	mp
			dP_1	(9, 25)	(48, 192)	p
			dP_2	(9, 24)	(46, 180)	m
			dP_2	(7, 21)	(40, 164)	m
1C2P1	4	$(2, 1, 1, 1)^{*o}$	dP_1	(6, 20)	(38, 164)	p
			dP_1	(6, 20)	(38, 164)	p
			dP_1	(6, 20)	(38, 164)	mp
			dP_1	(9, 25)	(48, 192)	p
			dP_1	(9, 25)	(48, 192)	p
1C2P2	1	$(2, 1, 1, 1)^{*o}$	dP_2	(8, 22)	(42, 168)	p
			dP_2	(5, 17)	(32, 140)	mp
			dP_2	(5, 17)	(32, 140)	p
			dP_1	(6, 20)	(38, 164)	p
			dP_1	(5, 18)	(34, 152)	mp
1C2P2	2	$(2, 1, 1, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_0	(6, 21)	(40, 176)	mp
			dP_1	(5, 18)	(34, 152)	mp
1C2P3	1	$(2, 1, 1, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_1	(3, 14)	(26, 128)	mp
			dP_1	(5, 18)	(34, 152)	mp

2 curves, 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Couplings
2C1P1	1	$(2, 1, 1, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_1	(5, 18)	(34, 152)	mp
			dP_2	(7, 21)	(40, 164)	m
			dP_1	(7, 22)	(42, 176)	m
2C1P1	2	$(2, 1, 1, 1)^{*o}$	dP_1	(5, 18)	(34, 152)	mp
			dP_2	(5, 17)	(32, 140)	mp
			dP_1	(7, 22)	(42, 176)	mp
			dP_2	(9, 25)	(48, 192)	p
			dP_2	(7, 21)	(40, 164)	m
2C1P2	1	$(2, 1, 1, 1)^{*o}$	dP_1	(9, 25)	(48, 192)	p
			dP_3	(6, 18)	(34, 140)	m
			dP_2	(6, 19)	(36, 152)	mp
			dP_0	(6, 21)	(40, 176)	mp
2C1P2	2	$(2, 1, 1, 1)^{*o}$	dP_1	(9, 25)	(48, 192)	p
			dP_2	(6, 19)	(36, 152)	mp
			dP_3	(6, 18)	(34, 140)	m
			dP_0	(6, 21)	(40, 176)	mp
2C1P2	3	$(2, 1, 1, 1)^{*o}$	dP_1	(6, 20)	(38, 164)	mp
			dP_3	(6, 18)	(34, 140)	m
			dP_1	(6, 20)	(38, 164)	p
			dP_1	(6, 20)	(38, 164)	p
2C1P2	4	$(2, 1, 1, 1)^{*o}$	dP_1	(9, 25)	(48, 192)	p
			dP_3	(6, 18)	(34, 140)	m
			dP_1	(6, 20)	(38, 164)	mp
			dP_1	(6, 20)	(38, 164)	p
			dP_1	(6, 20)	(38, 164)	p
2C1P3	1	$(2, 1, 1, 1)^{*o}$	dP_2	(6, 19)	(36, 152)	m
			dP_1	(4, 16)	(30, 140)	mp
			dP_1	(5, 18)	(34, 152)	mp
2C1P3	2	$(2, 1, 1, 1)^{*o}$	dP_1	(6, 20)	(38, 164)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_1	(5, 18)	(34, 152)	mp
2C1P3	3	$(2, 1, 1, 1)^{*o}$	dP_2	(8, 22)	(42, 168)	p
			dP_2	(6, 19)	(36, 152)	m
			dP_1	(4, 16)	(30, 140)	mp
			dP_2	(8, 22)	(42, 168)	p
			dP_1	(5, 18)	(34, 152)	mp
2C1P3	4	$(2, 1, 1, 1)^{*o}$	dP_2	(8, 22)	(42, 168)	p
			dP_1	(6, 20)	(38, 164)	mp
			dP_2	(4, 15)	(28, 128)	mp
			dP_2	(8, 22)	(42, 168)	p
			dP_1	(5, 18)	(34, 152)	m
2C1P4	1	$(2, 1, 1, 1)^{*o}$	dP_2	(5, 17)	(32, 140)	mp
			dP_2	(4, 16)	(30, 140)	mp
			dP_2	(5, 17)	(32, 140)	mp

Blowup of 1 point

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C1P1	1	$(2, 1)^{*o}$	dP_0	$(6, 21)$	$(40, 176)$	mp

Blowup of 2 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C2P1	1	$(2, 1, 1)^{*o}$	dP_0	$(6, 21)$	$(40, 176)$	mp
			dP_0	$(6, 21)$	$(40, 176)$	mp

Blowup of 3 points

Weights	Triang.	Base	dP	Genus	Yukawa	Decoupling
0C2P1	1	$(2, 1, 1, 1)^{*o}$	dP_0	$(6, 21)$	$(40, 176)$	p
			dP_0	$(6, 21)$	$(40, 176)$	p
			dP_0	$(6, 21)$	$(40, 176)$	p

B Fourfold data

Here we give the explicit data of the Calabi-Yau fourfolds constructed from the base manifolds of models 2, 3, 4 and 5. For convenience we relabeled the vertices obtained from the base. The vertex corresponding to the GUT divisor is given the coordinate w . The additional vertices/coordinates obtained after dualizing the reduced M-lattice polytope are denoted with a tilde. Furthermore we compute the Euler numbers for the $SO(10)$ model and compare with the formula (71).

B.1 Model 2

The vertices in the N-lattice are:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)$	2 2 2	x
	$\nu_2 = (-2 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1)$	3 3 3	y
	$\nu_3 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)$	1 0 0	z
	$\nu_4 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$	0 0 1	w
	$\nu_5 = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	0 1 0	y_1
∇_2	$\nu_6 = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	0 1 0	y_2
	$\nu_7 = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)$	0 1 0	y_3
	$\nu_8 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	0 1 1	y_4
	$\nu_9 = (0 \quad 0 \quad -1 \quad -1 \quad -1 \quad 1)$	0 1 0	y_5

(140)

After reducing the M-lattice polytope to the SO(10) case, we obtain for the dual N-lattice polytope:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)$	$2 \ 2 \ 1 \ 2 \ 0$	x
	$\nu_2 = (-2 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1)$	$3 \ 3 \ 2 \ 3 \ 0$	y
	$\nu_3 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)$	$1 \ 0 \ 0 \ 0 \ 0$	z
	$\nu_5 = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	$0 \ 1 \ 0 \ 0 \ 0$	y_1
	$\tilde{\nu}_{10} = (1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1)$	$0 \ 0 \ 1 \ 0 \ 0$	\tilde{y}_6
	$\tilde{\nu}_{11} = (0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0)$	$0 \ 0 \ 0 \ 0 \ 1$	\tilde{y}_7
	$\tilde{\nu}_{12} = (0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 1)$	$0 \ 0 \ 0 \ 1 \ 0$	\tilde{y}_8
∇_2	$\nu_6 = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	$0 \ 1 \ 0 \ 0 \ 0$	y_2
	$\nu_7 = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)$	$0 \ 1 \ 0 \ 0 \ 0$	y_3
	$\nu_8 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	$0 \ 1 \ 2 \ 2 \ 1$	y_4
	$\nu_9 = (0 \quad 0 \quad -1 \quad -1 \quad -1 \quad 1)$	$0 \ 1 \ 0 \ 0 \ 0$	y_5

(141)

The GUT vertex is again no longer a vertex but a point in ∇_1 . The Euler number is 1368 which matches with the result of the calculation using (71).

B.2 Model 3

We choose the following nef partition:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1)$	$2 \ 2 \ 2 \ 2$	x
	$\nu_2 = (-2 \quad 0 \quad 0 \quad -1 \quad 0 \quad -1)$	$3 \ 3 \ 3 \ 3$	y
	$\nu_3 = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1)$	$1 \ 0 \ 0 \ 0$	z
	$\nu_4 = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0)$	$0 \ 0 \ 1 \ 0$	w
	$\nu_5 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	$1 \ 0 \ 0 \ 0$	y_1
	$\nu_6 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$	$0 \ 0 \ 0 \ 1$	y_2
∇_2	$\nu_7 = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$	$1 \ 0 \ 0 \ 0$	y_3
	$\nu_8 = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	$1 \ 0 \ 1 \ 1$	y_4
	$\nu_9 = (0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1)$	$1 \ 0 \ 1 \ 0$	y_5
	$\nu_{10} = (0 \quad -1 \quad -1 \quad 0 \quad 1 \quad 0)$	$1 \ 0 \ 0 \ 0$	y_6

(142)

After reducing the M-lattice polytope to the $SO(10)$ case, we obtain for the dual N-lattice polytope:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \ 0 \ 0 \ 1 \ 0 \ 1)$	$2 \ 2 \ 1 \ 2 \ 1 \ 2 \ 0 \ 0$	x
	$\nu_2 = (-2 \ 0 \ 0 \ -1 \ 0 \ -1)$	$3 \ 3 \ 2 \ 3 \ 2 \ 3 \ 0 \ 0$	y
	$\nu_3 = (0 \ 0 \ 0 \ 1 \ 0 \ 1)$	$0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$	z
	$\nu_5 = (0 \ 1 \ 0 \ 0 \ 0 \ 0)$	$1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$	y_1
	$\tilde{\nu}_{11} = (0 \ 0 \ 0 \ -1 \ 0 \ 1)$	$0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0$	\tilde{y}_7
	$\tilde{\nu}_{12} = (1 \ 0 \ 0 \ -1 \ 2 \ 1)$	$0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0$	\tilde{y}_8
	$\tilde{\nu}_{12} = (1 \ 0 \ 0 \ -1 \ 0 \ 0)$	$0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1$	\tilde{y}_9
	$\tilde{\nu}_{13} = (0 \ 0 \ 0 \ -1 \ 1 \ -1)$	$0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0$	\tilde{y}_{10}
	$\tilde{\nu}_{14} = (0 \ 0 \ 0 \ -1 \ 0 \ 1)$	$0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0$	\tilde{y}_{11}
	$\tilde{\nu}_{15} = (0 \ 0 \ 0 \ -1 \ 2 \ -1)$	$0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0$	\tilde{y}_{12}
∇_2	$\nu_7 = (0 \ 0 \ 1 \ 0 \ 0 \ 0)$	$1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$	y_3
	$\nu_8 = (0 \ 0 \ 0 \ 1 \ 0 \ 0)$	$1 \ 0 \ 2 \ 2 \ 2 \ 2 \ 1 \ 1$	y_4
	$\nu_9 = (0 \ 0 \ 0 \ 0 \ -1 \ 1)$	$1 \ 0 \ 2 \ 2 \ 0 \ 0 \ 1 \ 0$	y_5
	$\nu_{10} = (0 \ -1 \ -1 \ 0 \ 1 \ 0)$	$1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$	y_6

(143)

Note that the divisors corresponding to the coordinates w (GUT divisor) and y_2 no longer appear as vertices of ∇_1 . However, they are still points. The Euler number is 960. Here we find a mismatch with the calculation using (71) where one obtains 552 as a result for the Euler number.

B.3 Model 4

The vertices in the N -lattice are:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \ 0 \ 1 \ 0 \ 0 \ 1)$	$2 \ 2 \ 2 \ 2 \ 2$	x
	$\nu_2 = (-2 \ 0 \ -1 \ 0 \ 0 \ -1)$	$3 \ 3 \ 3 \ 3 \ 3$	y
	$\nu_3 = (0 \ 0 \ 1 \ 0 \ 0 \ 1)$	$1 \ 0 \ 0 \ 0 \ 0$	z
	$\nu_4 = (0 \ 0 \ 0 \ 0 \ 1 \ 0)$	$0 \ 0 \ 1 \ 0 \ 0$	w
	$\nu_5 = (0 \ 1 \ 0 \ 0 \ 0 \ 0)$	$1 \ 0 \ 0 \ 0 \ 0$	y_1
	$\nu_6 = (0 \ 0 \ 0 \ 1 \ 0 \ 0)$	$0 \ 0 \ 1 \ 0 \ 0$	y_2
	$\nu_7 = (0 \ 0 \ 0 \ 0 \ 0 \ 1)$	$0 \ 0 \ 0 \ 0 \ 1$	y_3
∇_2	$\nu_8 = (0 \ 0 \ 1 \ 0 \ 0 \ 0)$	$1 \ 0 \ 1 \ 1 \ 1$	y_4
	$\nu_9 = (0 \ 0 \ 0 \ 0 \ -1 \ 1)$	$1 \ 0 \ 0 \ 1 \ 0$	y_5
	$\nu_{10} = (0 \ 0 \ 0 \ -1 \ 0 \ 1)$	$1 \ 0 \ 1 \ 0 \ 0$	y_6
	$\nu_{11} = (0 \ -1 \ 0 \ 1 \ 1 \ -1)$	$1 \ 0 \ 0 \ 0 \ 0$	y_7

(144)

After reducing the M-lattice polytope to the SO(10) case, we obtain for the dual N-lattice polytope:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \ 0 \ 1 \ 0 \ 0 \ 1)$	2 2 2 1 2 1 2 0 0	x
	$\nu_2 = (-2 \ 0 \ -1 \ 0 \ 0 \ -1)$	3 3 3 2 3 2 3 0 0	y
	$\nu_3 = (0 \ 0 \ 1 \ 0 \ 0 \ 1)$	0 1 0 0 0 0 0 0 0	z
	$\nu_5 = (0 \ 1 \ 0 \ 0 \ 0 \ 0)$	1 0 0 0 0 0 0 0 0	y_1
	$\nu_6 = (0 \ 0 \ 0 \ 1 \ 0 \ 0)$	0 0 1 0 0 0 0 0 0	y_2
	$\tilde{\nu}_{12} = (1 \ 0 \ -1 \ 0 \ 0 \ 1)$	0 0 0 0 0 1 0 0 0	\tilde{y}_8
	$\tilde{\nu}_{13} = (1 \ 0 \ -1 \ 0 \ 2 \ -1)$	0 0 0 0 1 0 0 0 0	\tilde{y}_9
	$\tilde{\nu}_{14} = (0 \ 0 \ -1 \ 0 \ 0 \ 0)$	0 0 0 0 0 0 0 0 1	\tilde{y}_{10}
	$\tilde{\nu}_{15} = (0 \ 0 \ -1 \ 0 \ 1 \ -1)$	0 0 0 0 0 0 0 1 0	\tilde{y}_{11}
	$\tilde{\nu}_{16} = (0 \ 0 \ -1 \ 0 \ 0 \ 1)$	0 0 0 0 0 0 1 0 0	\tilde{y}_{12}
	$\tilde{\nu}_{17} = (0 \ 0 \ -1 \ 0 \ 2 \ -1)$	0 0 0 0 1 0 0 0 0	\tilde{y}_{13}
∇_2	$\nu_8 = (0 \ 0 \ 1 \ 0 \ 0 \ 0)$	1 0 1 2 2 2 2 1 1	y_4
	$\nu_9 = (0 \ 0 \ 0 \ 0 \ -1 \ 1)$	1 0 0 2 2 0 0 1 0	y_5
	$\nu_{10} = (0 \ 0 \ 0 \ -1 \ 0 \ 1)$	1 0 1 0 0 0 0 0 0	y_6
	$\nu_{11} = (0 \ -1 \ 0 \ 1 \ 1 \ -1)$	1 0 0 0 0 0 0 0 0	y_7

(145)

The vertices corresponding to the GUT divisor and to the coordinate y_3 are no longer vertices but points in ∇_1 . The Euler number is 960. Using (71) to compute the Euler number we obtain 672 – so again, we find a mismatch.

B.4 Model 5

The vertices in the N-lattice are:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (3 \ 1 \ 1 \ 1 \ 0 \ 0)$	2 4 4 2 2	x
	$\nu_2 = (-2 \ -1 \ -1 \ -1 \ 0 \ 0)$	3 6 6 3 3	y
	$\nu_3 = (0 \ 1 \ 1 \ 1 \ 0 \ 0)$	0 0 0 0 1	z
	$\nu_4 = (0 \ 0 \ 0 \ 0 \ 1 \ 0)$	0 0 1 0 0	w
	$\nu_5 = (0 \ 0 \ 0 \ 0 \ 0 \ 1)$	1 0 0 0 0	y_1
	$\nu_6 = (0 \ -1 \ -1 \ 0 \ 1 \ 0)$	0 1 0 0 0	y_2
	$\nu_7 = (0 \ 1 \ 1 \ 1 \ -1 \ 1)$	0 1 1 0 0	y_3
	$\nu_8 = (0 \ 0 \ 0 \ 1 \ 0 \ 0)$	0 0 0 1 0	y_4
∇_2	$\nu_9 = (0 \ 1 \ 1 \ 1 \ 0 \ -1)$	1 1 1 0 0	y_5
	$\nu_{10} = (0 \ 0 \ 1 \ 0 \ 0 \ 0)$	0 1 0 1 0	y_6
	$\nu_{11} = (0 \ 1 \ 0 \ 0 \ 0 \ 0)$	0 1 0 1 0	y_7

(146)

After reducing the M-lattice polytope to the $SO(10)$ case, we obtain for the dual N-lattice polytope:

nef-part.	vertices	weights	coordinates
∇_1	$\nu_1 = (\quad 3 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0)$	2 4 6 5 2 2 2	x
	$\nu_2 = (-2 \quad -1 \quad -1 \quad -1 \quad 0 \quad 0)$	3 6 9 8 3 3 3	y
	$\nu_3 = (\quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0)$	0 0 0 0 0 0 1	z
	$\nu_5 = (\quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$	1 0 0 0 0 0 0	y_1
	$\nu_6 = (\quad 0 \quad -1 \quad -1 \quad 0 \quad 1 \quad 0)$	0 1 0 0 0 0 0	y_2
	$\nu_7 = (\quad 0 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1)$	0 1 2 2 1 0 0	y_3
	$\nu_8 = (\quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$	0 0 0 0 0 1 0	y_4
	$\tilde{\nu}_{12} = (\quad 0 \quad -1 \quad -1 \quad -1 \quad 2 \quad 0)$	0 0 1 0 0 0 0	\tilde{y}_8
	$\tilde{\nu}_{13} = (\quad 1 \quad -1 \quad -1 \quad -1 \quad 2 \quad 0)$	0 0 0 1 0 0 0	\tilde{y}_9
	$\tilde{\nu}_{14} = (\quad 0 \quad -1 \quad -1 \quad -1 \quad 1 \quad 0)$	0 0 0 0 1 0 0	\tilde{y}_{10}
∇_2	$\nu_9 = (\quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad -1)$	1 1 2 2 1 0 0	y_5
	$\nu_{10} = (\quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$	0 1 0 0 0 1 0	y_6
	$\nu_{11} = (\quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$	0 1 0 0 0 1 0	y_7

(147)

The GUT divisor is no longer a vertex but a point in ∇_1 . The Euler number is 4872. This coincides with the result one gets from (71).

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